1) So far -

we've defined relative Fargues-Fontaine curves over perfectoid spaces.

$$\begin{array}{c} FF_S \\ \vdots \\ \downarrow \\ S \end{array} \qquad \left(\begin{array}{c} FF_S^{\diamond} \\ \downarrow \\ S^{\diamond} \end{array} \right)$$

over a point $\mathrm{Spa}(\mathbb{F}, O_F)$

vector bundles on FF_S "capture" p-adic representations of G_F.

(generalizes over a base)

suggests to study <u>moduli spaces</u> of vector bundles on FF_*.

(must work in certain categories of <u>stacks</u>)

2)

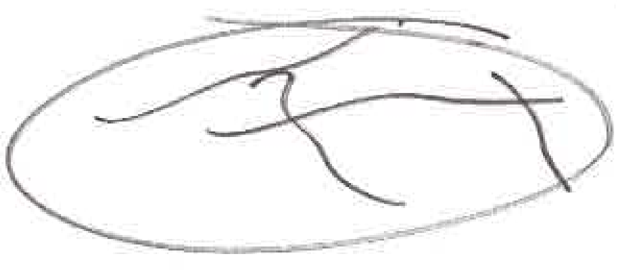
By analogy with geometric invariant theory (GIT), we will ~~stratf~~ stratify moduli spaces by "Newton polygons"

|

help understand behavior of period mappings

Gross-Hopkins
Rapoport-Zink
Scholze-Weinstein
Caraiani-Scholze (Hodge-Tate)

typical phenomenon: see étale local system on image of period mappins (open)



3)
which can be extended as a
<u>vector bundle</u>.

— Scholze, Fargues: "shtukas"
use these ideas to build objects
analogous to those used in
geometric Langlands (Drinfeld,
L. Lafforgue,
V. Lafforgue
local Genestier)
w/plan of realizing Langlands
correspondence (for p-adic fields)
in cohomology of moduli stacks
of these objects.

— this ties back nicely to existing
concepts in p-adic Hodge theory,
e.g. crystalline representations.

4)

$S = \mathrm{Spa}(F, \mathcal{O}_F)$ F = alg. closed perfectoid field.

Thm from last time on Galois representations

$\Uparrow$

classification theorem for vector bundles

<u>Thm</u> $\binom{K}{F-F}$ Every vector bundle on FF_S splits as a direct sum of $\mathcal{O}(d)$ for $d \in \mathbb{Q}$.

these are building blocks of Dieudonné-Manin of classification of isocrystals over an alg. closed field.

$d = \frac{r}{s}$ $\mathcal{O}(d) \rightsquigarrow$ rank s φ-equivariant vector bundle on Y_S

$(r, s) = 1$ $\varphi^* \mathcal{O}(d) \simeq \mathcal{O}(d)$

$1 \otimes v_1 \longrightarrow v_2$

$\vdots$

$\longrightarrow v_s$

$1 \otimes v_s \longrightarrow p^{-r} v_1$

$$\begin{array}{c} Y_S \\ \downarrow \\ FF_S \end{array}$$

$= Y_S/\varphi.$

Can view this bundle $\mathcal{O}(d)$ as pushforward of rank 1 bundle on a degree-s ~~unramified~~ étale cover of X_S.

(i.e. $Y_S/\varphi^{s\mathbb{Z}}$)

$\mathcal{O}(d)$ is (semi)stable of ~~deg~~ slope d.

$\mathrm{Spa}(F, \mathcal{O}_F)$ F general perfectoid field of char p.

Every vector bundle V on FF_S has a unique filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$$

where each V_i/V_{i-1} is semistable vector bundles of slope μ_i

and $\mu_1 > \cdots > \mu_\ell$.

V_1 = maximal destabilizing subbundle.

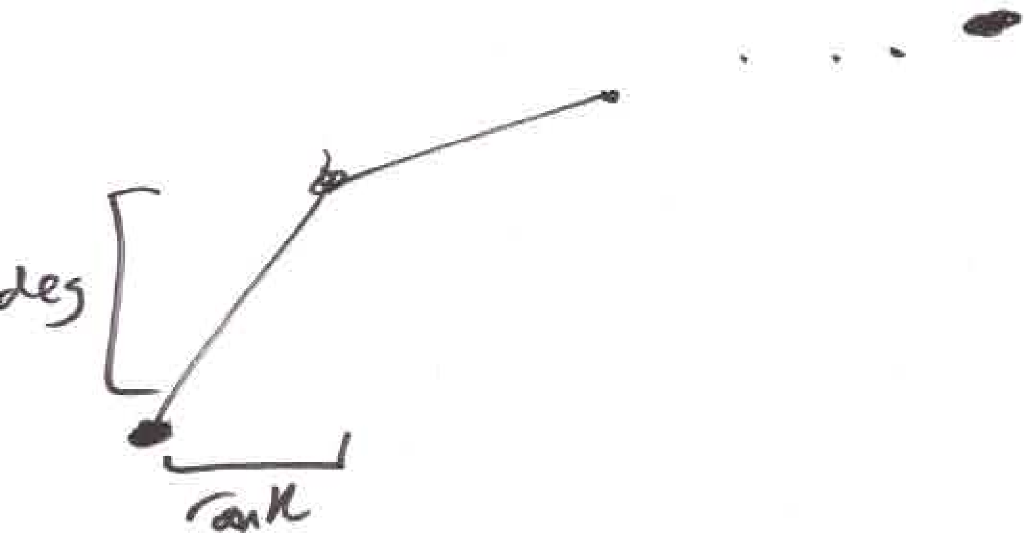
Harder–Narasimhan (<u>HN filtrations</u>)

poor

II) associate to V a "Newton polygon"

6

where slopes are μ_i
with multiplicity rank (V_i/V_{i-1})



How can slopes interact?
e.g. in an SES
$$0 \to V_1 \to V \to V_2 \to 0$$
what are the possible polygons associated to the three terms?

7) Let S be any ~~petctoid~~ perfectoid space in char p.

Let V be a vector bundle ~~of~~ on FF_S.

<u>Thm</u> (K-L)

- rank is locally constant
- degree is locally constant.

- HN polygon, as a function on S, is <u>upper semicontinuous</u>.

  (e.g. the semistable locus is <u>open</u> subspace of S).

  (in general, locally closed stratification by polygons).

- if V is everywhere semistable of degree 0 (fiberwise)

then V corresponds to a unique (pro-)étale $\mathbb{Q}_p$-local system

(a ~~sh~~ pro-étale sheaf of $\mathbb{Q}_p$-modules which are locally finite free).

proétale

8)

$$\begin{array}{c} \mathrm{Com.} \\ \big\downarrow z. \\ E \end{array}$$

Motivation for Drinfeld's shtukas.
is "Drinfeld's lemma"

if $k = \bar{k}$ is a field of char 0,

then $\pi_1^{\text{prof}}(X \times_k Y) \cong \pi_1^{\text{prof}}(X) \times \pi_1^{\text{prof}}(Y)$.

(under some hypotheses on X, Y).

this fails in char p

e.g. $X = Y = \mathbb{A}^1_k$.

but...

9) ...

For X, Y perfect connected schemes $/ \mathbb{F}_p$

$$\pi_1^{prof}(X) \times \pi_1^{prof}(Y) \cong$$

~~ÉE~~ finite étale covers of Y

$$\pi_1^{prof}\left(X \times_{\mathbb{F}_p} \frac{Y}{\varphi_Y} \right) \quad \varphi_Y\text{-equivariant finite étale covers.}$$

$$\cong \pi_1^{prof}\left(\frac{X}{\varphi} \times_{\mathbb{F}_p} Y \right)$$

analogue for perfectoid spaces

(Scholze).

10). Drinfeld's shtukas (for $C/\mathbb{F}_p$ curve)

over a base S =

vector bundle V on $C \times_{\mathbb{F}_p} S$

plus not-quite-an-isomorphism

$$\varphi_S^* V \longrightarrow V$$

allow poles along some sections of

$$\begin{array}{c} C\times S \\ \uparrow\downarrow \\ S \end{array}$$

<u>less</u> of shtukas

11) Similarly, in analytic world

want $C \times S \rightsquigarrow \mathrm{Spd}(\mathbb{Z}_p) \times S$

perfectoid. inclusion

??

$\mathrm{Spd}(\mathbb{Q}_p) \times S$

$\mathrm{Spd}(\mathrm{Spa}(A_{\inf}) - \{[\varpi] = 0\})^{\diamond} = \underline{\mathrm{Spd}(Y_S)^{\diamond}}$

sections "less"

$S \longrightarrow \mathrm{Spd}(\mathbb{Z}_p)$

i.e. ~~untilts~~ untilts!!