1)

Perfectoid rings and spaces

(A, A^+) = Huber pair
(A complete, analytic)

(A, A^+) is <u>perfectoid</u> if
(analytic: topologically nilpotent elements generate unit ideal)

- A is <u>uniform</u> (A° is bounded)
  (uniform + analytic $\Rightarrow$ A^+ is also a ring of definition)
- there exists an ideal of definition I of A^+ s.t. $p \in I^p$
- for some such I,
  Frobenius: $A^+/I \to A^+/I^p$
  is <u>surjective</u>.

Comments:
- Any perfectoid field A has this property
- This depends only on A, not on A^+

2) comments continued:

- if $p=0$ in A then

A perfectoid $\iff$ A uniform, perfect (and analytic)

- if A is Tate, can take $I = (\varpi)$ for a suitable pseudouniformizer ϖ.

(if A is a $\mathbb{Q}_p$-algebra, then A is Tate)

there do exist "natural" examples of perfectoid rings which are Tate but not $\mathbb{Q}_p$-algebras (next time)

- A perfectoid $\Rightarrow$ A sheafy so can define <u>perfectoid spaces</u> as adic spaces built out of $\mathrm{Spa}(A, A^+)$ where A is perfectoid.

3)

Examples

- Any perfectoid field is a perfectoid ring.

  (e.g. $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, $\widehat{\mathbb{Q}_p(p^{p^{-\infty}})}$

  $\mathbb{C}_p$, completion of any totally ramified, p-adic Lie extension of $K/\mathbb{Q}_p$ finite $\Leftarrow$ Fontaine-Wintenberger Sen)

- if A is a perfectoid ring, then so are $A\langle T^{p^{-\infty}}\rangle$ $\quad I \subset A^+$, ideal of definition

  $$= \left(A^+[T, T^{1/p}, T^{1/p^2}, \dots]^{\wedge}_{I}\right) \otimes_{A^+} A$$

  and $A\langle T^{\pm p^{-\infty}}\rangle$

  (or weighted analogues)

4)

Examples of perfectoid spaces

- ~~K~~ K = perfectoid field

$$\mathbb{P} \xleftarrow{\lim} \mathbb{P}^{n,an}_K$$

$$[x_0 : \cdots x_n] \to [x_0^p : \cdots : x_n^p]$$

General theme: taking an inverse limit of adic spaces which "should be inseparable in characteristic p" often leads to a perfectoid space.

Similarly with $\mathbb{P}^n$ with a toric variety

- $\varprojlim E^{an}$ or $\varprojlim A^{an}$

elliptic curve   analytic variety

over a perfectoid field

via $[m]$ $m > 0$, $p \mid m$

is perfectoid.

5)

- $\varprojlim_{k} X(Np^k)$ tower of modular curves over a perfectoid field

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Tilting and untilting:

Let (A, A^+) be a perfectoid pair.

$A^\flat := \varprojlim_{x \mapsto x^p} A$ as a multiplicative monoid

$\cup$

$A^{\flat +} := \varprojlim_{x \mapsto x^p} A^+$

<u>Thm</u> The formula for addition

$(x_n)_n + (y_n)_n = (z_n)_n,$

$z_n := \lim_{m \to \infty} (x_{m+n} + y_{m+n})^{p^m}$

equips $A^\flat$ ~~with~~ with a ring structure and...

$(A^\flat, A^{\flat +})$ forms a perfectoid pair of characteristic p.

6)

Note: $A^{b+} := \varprojlim_{x \to x^p} A^+ \cong \varprojlim_{x \to x^p} A^+/(I)$

$\#: A^b \to A^\#$ $\quad (x_n)_n \longrightarrow x_0$

$A^{b+} \to A^+$ is continuous. & multiplicative.

The tilting functor

$(A, A^+) \longrightarrow (A^b, A^{b+})$

$\{\text{perfectoid pairs}\} \to \{\text{perfectoid pairs of characteristic } p\}$

is not an equivalence.

but becomes an equivalence if you keep track of extra data on the char p side.

$\theta: W(A^{b+}) \twoheadrightarrow A^+$ surjective.

$\sum p^n [\bar{x}_n] \longrightarrow \sum p^n \#(\bar{x}_n)$

7)

Kernel of θ is generated by an element z which is primitive (of degree 1)

$$z = \sum p^n [\bar{z}_n] \in W(A^{b+})$$

is primitive if

$\bar{z}_0$ = topologically nilpotent

$\bar{z}_1$ = unit in A^{b+}

$\Updownarrow$

$$z = [\bar{z}_0] + p z_1$$

z_1 is a unit in $W(A^{b+})$.

(like a monic polynomial in p or better, a Weierstrass poly in p of degree 1)

$\Downarrow$

comparison with $\mathbb{Z}_p[\![T]\!] \ni T-p$
have Euclidean division

8) Note:

any multple of a primitive element by a unit is still primitive

In fact,
generators of $\ker(\theta)$
are precisely the primitive elements in kernel.

<u>Thm</u> The functor

$(A, A^+) \longrightarrow (A^{b\flat}, A^{b+}, \ker(\theta))$

$\{$ perfectoid pair $\}$ $\{$ triples (R, R^+, I) where (R, R^+) is a perfectoid pair of char p and $I \subseteq W(R^+)$ is an ideal generated by some primitive element of degree 1 $\}$

is an equivalence of categories, with quasi-inverse

$(R, R^+, I) \to (\ \ldots \ , W(R^+)/I)$

9) in general,

$$\cdots = W^{bd}(R)/(I)$$

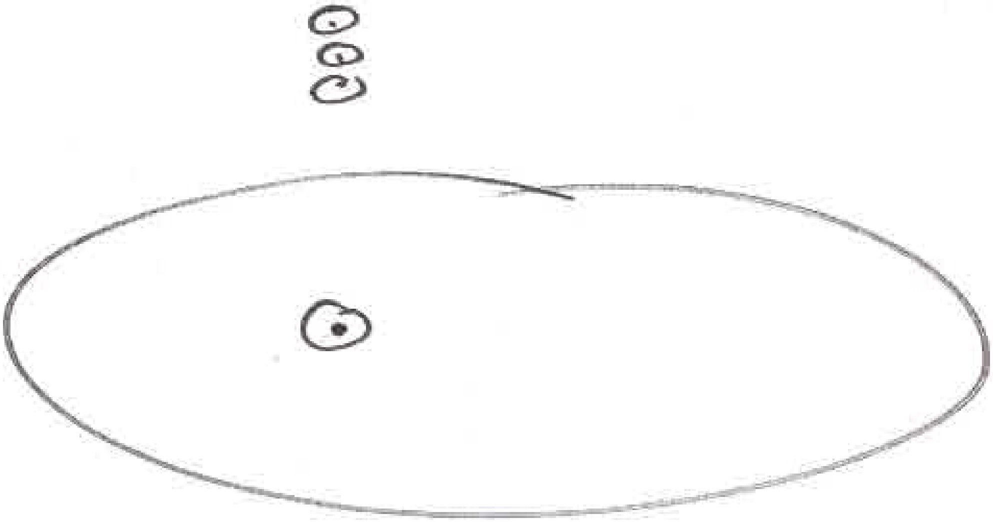
<u>bounded</u> Witt vectors over R.

for perfectoid pairs with $A/\mathbb{Q}_p$,
can instead $W(R^+)(p^{-1})/I$.

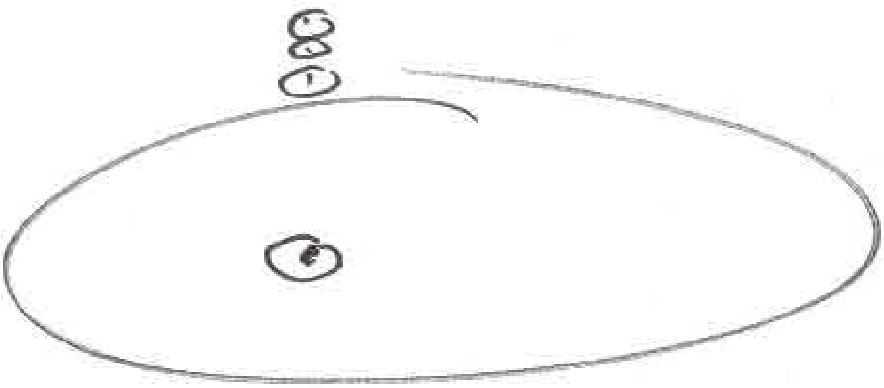
This equivalence preserves many additional properties:

- $\mathrm{Spa}(A, A^+) \cong \mathrm{Spa}(A^\flat, A^{\flat +})$ matching rational subspaces
- rational localizations match up.
  $\Rightarrow$ stably uniform $\Rightarrow$ sheafy $\Rightarrow$ acyclic
- finite étale algebras match up.
  (reduces to field case using henselian property of adic local rings)
  analytic

10)



$\mathrm{Spa}(A, A^+)$



$\mathrm{Spa}(A^\flat, A^{\flat +})$

$$\Downarrow$$

Faltings almost purity theorem