PERIOD RINGS AND PERIOD SHEAVES

1. Background Story

The starting point of p-adic Hodge theory is the comparison conjectures (now theorems) between p-adic étale cohomology, de Rham cohomology, and (log-)crystalline cohomology.

Throughout the notes, K is a finite extension of $\mathbb{Q}_p$ with residue field k. Let $W(k)$ be the Witt vectors with coefficients in k and let $K_0 = \operatorname{Frac} W(k)$.

Theorem 1.1. *(Hodge-Tate comparison)*
Let X be a proper smooth variety over K. There exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{0 \le i \le n} H^{n-i}(X, \Omega^i_{X/K}) \otimes_K \mathbb{C}_p(-i)$$

compatible with G_K-actions.

Theorem 1.2. *(de Rham comparison)*
Let X be a proper smooth variety over K. There exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^n_{\text{dR}}(X/K) \otimes_K B_{\text{dR}}$$

compatible with G_K-actions and filtrations.

Theorem 1.3. *(crystalline comparison)*
Let X be a proper smooth variety over K. Suppose X has a proper smooth model $\mathfrak{X}$ over $\mathcal{O}_K$. Let X_0 denote the special fiber of $\mathfrak{X}$. There exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \cong H^n_{\text{crys}}(X_0/W(k)) \otimes_{W(k)} B_{\text{crys}}$$

compatible with G_K-actions, Frobenius actions, and filtrations.

There is also a semistable comparison relating p-adic étale cohomology to log-crystalline cohomology.

The purpose of these notes is to construct and study various *period rings* including B_{dR} and B_{crys}. Good references for an introduction to p-adic Hodge theory include [BC], [FO], and [Ber]. The problems and examples here are by no means original, most of which are inspired from literatures mentioned above.

2. Witt Vectors

Definition 2.1. Let A be a topological ring and let $A \supset I_1 \supset I_2 \supset \cdots$ be a decreasing chain of ideals. Assume A/I_1 is an $\mathbb{F}_p$-algebra and $I_n \cdot I_m \subset I_{n+m}$. The topology on A is given by $(I_n)_{n \ge 1}$.

(i) A is called a *p-ring* if the topology is separated and completed.

(ii) A is called a *strict p-ring* if moreover $I_n = p^n A$ and p is not a zero divisor in A.

Let A be a p-ring with perfect residue ring $R = A/I_1$. For $x \in R$ and $n \in \mathbb{N}$, let $x_n = x^{1/p^n}$. Let $\widehat{x}_n$ be a lift of x_n to A. The *Teichmüller lift* of x is defined to be

$$[x] := \lim_n (\widehat{x}_n)^{p^n}$$

In particular, if A is a strict p-ring with perfect residue ring R, then every $a \in A$ can be uniquely written as

$$a = \sum_{n=0}^{\infty} p^n [a_n]$$

with $a_n \in R$. (see Problem 2)

Theorem 2.2. *If R is a perfect ring of characteristic p. Then there exists a unique strict p-ring $W(R)$ with residue ring R.*

Roughly speaking, $W(R) = \{\text{“}\sum_{n=0}^{\infty} p^n [x_n]\text{”} \mid x_n \in R\}$. For an explicit description of $W(R)$, see Problem 4.

Theorem 2.3. *(Universality)*
Let R_0 be a perfect ring of characteristic p. Let A be any p-ring with residue ring R. Suppose $\alpha : R_0 \to R$ is a ring homomorphism and $\widetilde{\alpha} : R_0 \to A$ is a multiplicative lift of α, then there exists a unique homomorphism $\alpha : W(R_0) \to A$ such that $\alpha([x]) = \widetilde{\alpha}(x)$.

Remark 2.4. Witt vectors can be defined for more general rings, not necessarily $\mathbb{F}_p$-algebras. For details, we refer to [Ser].

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Problem 1. Which of the following rings are p-rings? Strict p-rings?

(a) $\mathcal{O}_K$ (where $K/\mathbb{Q}_p$ is a finite extension.)
(b) $\mathcal{O}_{\overline{K}}$
(c) $\mathcal{O}_{\mathbb{C}_p}$
(d) $\mathcal{O}_K[[X_J^{1/p^\infty}]] = \varprojlim_n (\cup_{m=0}^{\infty} \mathcal{O}_K[X_j^{1/p^m}; j \in J])/p^n$
(J is any index set).
(e) R^+ (where (R, R^+) is a perfectoid algebra of characteristic 0).

Problem 2. Let A be a strict p-ring with perfect residue ring R. Show that every $a \in A$ can be uniquely written as

$$a = \sum_{n=0}^{\infty} p^n [a_n]$$

with $a_n \in R$.

Problem 3. (Universal Witt polynomials)
Consider strict p-ring $S = \mathbb{Z}_p[[X_i^{1/p^\infty}, Y_i^{1/p^\infty}]]_{i\geq 0}$ with residue ring $\overline{S} = \mathbb{F}_p[X_i^{1/p^\infty}, Y_i^{1/p^\infty}]_{i\geq 0}$.

(i) Show that there exist polynomials P_i, $Q_i \in \overline{S}$ such that

$$\sum_{i=0}^{\infty} p^i[X_i] + \sum_{i=0}^{\infty} p^i[Y_i] = \sum_{i=0}^{\infty} p^i[P_i]$$

$$\Big(\sum_{i=0}^{\infty} p^i[X_i]\Big)\Big(\sum_{i=0}^{\infty} p^i[Y_i]\Big) = \sum_{i=0}^{\infty} p^i[Q_i]$$

(ii) Calculate P_0, P_1, Q_0, Q_1.

(iii) Show that P_i's and Q_i's are universal in the following sense. For any strict p-ring A with perfect residue ring R and any $x_0, x_1, \ldots, y_0, y_1, \ldots \in R$, we have

$$\sum_{i=0}^{\infty} p^i[x_i] + \sum_{i=0}^{\infty} p^i[y_i] = \sum_{i=0}^{\infty} p^i[P_i(x_0, x_1, \ldots, y_0, y_1, \ldots)]$$

$$\Big(\sum_{i=0}^{\infty} p^i[x_i]\Big)\Big(\sum_{i=0}^{\infty} p^i[y_i]\Big) = \sum_{i=0}^{\infty} p^i[Q_i(x_0, x_1, \ldots, y_0, y_1, \ldots)]$$

Problem 4. (Explicit description of $W(R)$)
Let R be a perfect ring of characteristic p. Suppose R has a presentation $R \cong \overline{S}_J/I$ where

$$\overline{S}_J = \mathbb{F}_p[X_J^{1/p^\infty}]$$

for some index set J, and I is a perfect ideal of $\overline{S}_J$.

(i) Show that such a presentation always exists.

(ii) Consider

$$S_J := \mathbb{Z}_p[[X_J^{1/p^\infty}]]$$

Show that $W(R) \cong S_J/W(I)$ where

$$W(I) = \Big\{ \sum_{i=0}^{\infty} p^i[x_i] \,|\, x_i \in I \Big\}.$$

(iii) Prove Theorem 2.3 using the explicit description above.

Problem 5.($\mathcal{O}_{\mathbb{C}_p^\flat}$ and $W(\mathcal{O}_{\mathbb{C}_p^\flat})$)
Let $(\mathbb{C}_p^\flat, \mathcal{O}_{\mathbb{C}_p^\flat})$ be the tilt of the perfectoid field $(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ (see [Sch1]). We briefly review the construction here. Consider

$$\mathcal{O}_{\mathbb{C}_p^\flat} := \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p$$

equipped with inverse limit topology. This is a perfect ring of characteristic p. Let $\mathbb{C}_p^\flat = \operatorname{Frac} \mathcal{O}_{\mathbb{C}_p^\flat}$. The natural projection gives a multiplicative homeomorphism

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p.$$

The inverse is given by

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p \to \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$$

sending $x = (x_0, x_1, \ldots)$ to $(x^{\#}, x^{\#(1)}, x^{\#(2)}, \ldots)$ where $x^{\#(m)} = \lim_{n\to\infty} \widehat{x}_n^{p^{n-m}}$. One can define a valuation on $\mathcal{O}_{\mathbb{C}_p^\flat}$ by $|x| := |x^{\#}|_{\mathbb{C}_p}$.

(i) Check that $|\cdot|$ is indeed a non-archimedean valuation on $\mathcal{O}_{\mathbb{C}_p^\flat}$. In particular, $|x+y| \le \max(|x|, |y|)$, $\forall\ x, y \in \mathcal{O}_{\mathbb{C}_p^\flat}$.

(ii) Check that $\mathcal{O}_{\mathbb{C}_p^\flat}$ is complete and separated with respect to $|\cdot|$.

(iii) Consider

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\mathbb{C}_p^\flat}$$

For each $n \in \mathbb{N}$, calculate $|\varepsilon^{1/p^n} - 1|$.

(iv) Let $K/\mathbb{Q}_p$ be a finite extension and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Then $\mathcal{O}_{\mathbb{C}_p^\flat}$ is equipped with a natural Frobenius action φ and an action of G_K. More precisely, for $x = (x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\mathbb{C}_p^\flat}$, we define

$$\varphi(x) = ((x^{(0)})^p, (x^{(1)})^p, \ldots)$$

and

$$g(x) = (g(x^{(0)}), g(x^{(1)}), \ldots), \ \ \forall g \in G_K.$$

The φ and G_K actions also extend naturally to $W(\mathcal{O}_{\mathbb{C}_p^\flat})$.

Find $(\mathcal{O}_{\mathbb{C}_p^\flat})^{\varphi=1}$, $(\mathcal{O}_{\mathbb{C}_p^\flat})^{G_K}$, $W(\mathcal{O}_{\mathbb{C}_p^\flat})^{\varphi=1}$, $W(\mathcal{O}_{\mathbb{C}_p^\flat})^{G_K}$.

3. De Rham period ring B_{dR}

Consider the following G_K-equivariant ring homomorphism

$$\theta : W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p}$$

$$\sum_{i=0}^{\infty} p^i [x_i] \mapsto \sum_{i=0}^{\infty} p^i x_i^{\#}$$

It turns out $\ker(\theta)$ is a principle ideal generated by $\xi = [p^\flat] - p$, where

$$p^\flat = (p, p^{1/p}, p^{1/p^2}, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\mathbb{C}_p^\flat}.$$

Define B_{dR}^+ to be the $\ker(\theta)$-adic completion of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$; i.e.,

$$B_{\mathrm{dR}}^+ = \varprojlim_n W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]/(\ker\theta)^n.$$

The natural projection induces

$$\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \to W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]/(\ker\theta) \cong \mathbb{C}_p.$$

In particular, B_{dR}^+ is a complete discrete valuation ring with maximal ideal $\mathfrak{m}_{B_{\mathrm{dR}}^+} = (\ker\theta)$ and residue field $\mathbb{C}_p$. We temporarily equip B_{dR}^+ with the discrete valuation ring topology. (See Problem 11 for a "better" topology.)

Let ε be the same as in Problem 5(iii). Consider the element

$$t = \log[\varepsilon] = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n}.$$

One can check that t converges to a uniformizer in B_{dR}^+. Moreover, t has the nice property that

$$g(t) = \chi(g)t, \quad \forall g \in G_K$$

where χ is the cyclotomic character.

Finally, we define $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\frac{1}{t}] = \operatorname{Frac} B_{\mathrm{dR}}^+$, which carries a natural G_K-action. One can prove $B_{\mathrm{dR}}^{G_K} = K$. In addition, one can put a G_K-stable filtration on B_{dR} by setting

$$\operatorname{Fil}^n B_{\mathrm{dR}} := t^n B_{\mathrm{dR}}^+ = \mathfrak{m}_{B_{\mathrm{dR}}^+}^n, \quad n \in \mathbb{Z}.$$

However, the φ-action on $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ does not extend to B_{dR}.

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Problem 6. If we identify $W(\mathcal{O}_{\mathbb{C}_p^\flat})$ with $(\mathcal{O}_{\mathbb{C}_p^\flat})^{\mathbb{N}}$ and equip the product of valuation topology from $\mathcal{O}_{\mathbb{C}_p^\flat}$, show that $\theta : W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p}$ is open.

Problem 7. Let k be the residue field of K. Show that θ is actually a morphism of $W(\bar{k})$-algebras with the natural $W(\bar{k})$-structures on both sides.

Problem 8. For $\alpha \in W(\mathcal{O}_{\mathbb{C}_p^\flat})$, let $\overline{\alpha}$ denote the reduction of α mod p. Show that $\alpha \in \ker(\theta)$ is a generator if and only if $|\overline{\alpha}| = 1$. In particular, ξ is a generator.

Problem 9. Show that φ-action on $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ does not extend to B_{dR}^+.

Problem 10. Show that $[p^\flat]$ is invertible in B_{dR}^+.

Problem 11. This is a famous exercise in [BC]. We put a new topological ring structure on $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ which extends to one on B_{dR}^+ such that the quotient topology on $\mathbb{C}_p$ through θ_{dR}^+ is the natural valuation topology!

(i) For any open ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}_p^\flat}$ and $N \geq 0$, consider

$$U_{N,\mathfrak{a}} := \bigcup_{j > -N} \left(p^{-j} W(\mathfrak{a}^{p^j}) + p^N W(\mathcal{O}_{\mathbb{C}_p^\flat})\right) \subset W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}].$$

Prove that $U_{N,\mathfrak{a}}$ is a G_K-stable $W(\mathcal{O}_{\mathbb{C}_p^\flat})$-submodule of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$.

(ii) Define a topological ring structure on $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ by making $U_{N,\mathfrak{a}}$'s a base of open neighborhoods of 0. Show that the topological ring structure is well-defined and the G_K-action is continuous under this topology.

(iii) Show that $\theta : W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}] \to \mathbb{C}_p$ is continuous and open, where $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ is equipped with the new topology and $\mathbb{C}_p$ with valuation topology.
(iv) Show that $(\ker\theta)^n = \xi^n W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$ are closed ideals of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]$.
(v) Equip B_{dR}^+ with the inverse limit topology of the quotient topologies on each $W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}]/(\ker\theta)^n$. Verify that the quotient topology on $\mathbb{C}_p$ through $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \to \mathbb{C}_p$ coincides with the valuation topology.
(vi) Show that the new topology on B_{dR}^+ is complete.

Problem 12. Prove that $g(t) = \chi(g)t$ for all $g \in G_K$.
(Hint: Show that both sides are equal to "$\log([\varepsilon]^{\chi(g)})$". This expression does not converge in discrete valuation topology, but converges in the new topology constructed in Problem 11.)

Problem 13.(G_K-cohomology of B_{dR})
(i) Calculate $H^i(G_K, t^j B_{\mathrm{dR}}^+)$ for $i = 0, 1$ and for all $j \geq 1$.
(ii) Calculate $(B_{\mathrm{dR}})^{G_K}$ and $(B_{\mathrm{dR}}^+)^{G_K}$.

4. De Rham representations

Let $K/\mathbb{Q}_p$ be a finite extension and let $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ denote the category of G_K-representations; i.e., finite dimensional $\mathbb{Q}_p$-vector spaces with a continuous action of G_K. Let Fil_K denote the category of *filtered K-vector spaces*; i.e., finite dimensional K-vector spaces D equipped with an *exhaustive* and *separated* filtration $\{\mathrm{Fil}^i(D)\}_{i\in\mathbb{Z}}$. Being exhaustive means $\mathrm{Fil}^i(D) = D$ for $i \ll 0$, and being separated means $\mathrm{Fil}^i(D) = 0$ for $i \gg 0$.

Consider functor

$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Fil}_K$$
$$V \mapsto (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$$

The filtration on $D_{\mathrm{dR}}(V)$ is given by $\mathrm{Fil}^i(D_{\mathrm{dR}}(V)) := (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K}$.

It is always true that $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$. We say V is *de Rham* if this is an equality. The subcategory of de Rham representations is denoted by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$.

Theorem 4.1. (i) *The functor*

$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Fil}_K$$

is exact and faithful (but not full!) Moreover, it respects direct sums, tensor products, subobjects, quotients, and duals.

(ii) *If V is de Rham, the natural map*

$$\alpha_{\mathrm{dR},V} : D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

is an isomorphism of filtered vector spaces.

The notion of *Hodge-Tate* representations can be defined in the same fashion. The *Hodge-Tate period ring* is defined to be

$$B_{\mathrm{HT}} = \oplus_{n\in\mathbb{Z}}\mathbb{C}_p(n)$$

where $\mathbb{C}_p(n)$ standards for the Tate twist. For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we can consider functor

$$D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vect}_K.$$

A representation V is called *Hodge-Tate* if $\dim_K D_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V$.

Theorem 4.2. *De Rham representations are Hodge-Tate.*

Important source of de Rham representations: those G_K-representations arising from p-adic étale cohomologies of proper smooth varieties over K are de Rham.

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Problem 14. Prove Theorem 4.2.

Problem 15. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and let $n \in \mathbb{Z}$. Prove that V is de Rham if and only if $V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ is de Rham.

Problem 16. Let K'/K be a finite extension and let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Show that V is de Rham as a G_K-representation if and only if it is de Rham viewed as a $G_{K'}$-representations.

Problem 17. Suppose $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is 1-dimensional. Prove that V is de Rham if and only if it is Hodge-Tate.

Problem 18. Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character. Show that $\mathbb{Q}_p(\eta)$ is de Rham (equivalently, Hodge-Tate) if and only if there exists $n \in \mathbb{Z}$ such that $\chi^n\eta$ is *potentially unramified*. (A character of G_K is called *potentially unramified* if there exists a finite extension L/K such that the image of I_L is trivial.)

Problem 19. (Tate curve)
Let $K/\mathbb{Q}_p$ be a finite extension and let $q \in K^\times$ be an element such that $|q| < 1$. Let $q^{\mathbb{Z}} = \{q^n \mid n \in \mathbb{Z}\}$ and consider quotient group

$$E_q = \overline{K}^\times/q^{\mathbb{Z}} \quad (\text{“Tate curve”})$$

The abelian group E_q has a natural action of G_K. For each $n \geq 0$, let $E_q[p^n]$ be the subgroup of p^n-torsion elements. Define the *Tate module*

$$T_p(E_q) := \varprojlim_n E_q[p^n]$$

with transition maps being multiplication by p. Inverting p, we obtain the *rational Tate module*

$$V_p(E_q) := T_p(E_q) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(i) For each n, choose a primitive p^n-th root of unity ζ_{p^n} and choose a p^n-th root λ_n of q in $\overline{K}^\times$. Show that

$$(\mathbb{Z}/p^n\mathbb{Z})^2 \to E_q[p^n]$$
$$(a,b) \mapsto \zeta_{p^n}^a \lambda_n^b$$

is an isomorphism.

(ii) Show that $V_p(E_q)$ is a 2-dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of G_K.

(iii) Show that $V_p(E_q)$ is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$.

$$0 \to \mathbb{Q}_p(1) \to V_p(E_q) \to \mathbb{Q}_p \to 0$$

(iv) $V_p(E_q)$ has an explicit basis $\{e, f\}$ where

$$e = (1, \zeta_p, \zeta_{p^2}, \ldots), \quad f = (q, q^{1/p}, q^{1/p^2}, \ldots).$$

For any $g \in G_K$, show that $g(e) = \chi(g)e$, $g(f) = f + a(g)e$ for some $a(g) \in \mathbb{Z}_p$ depending on g.

(v) Recall that $t = \log[\varepsilon] \in B_{\mathrm{dR}}^+$. Let $q^\flat = (q, q^{1/p}, \ldots) \in \mathcal{O}_{\mathbb{C}_p^\flat}$. We can define "$\log[q^\flat]$" as follows.

$$\log[q^\flat] := \log_p(q) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([q^\flat]/q - 1)^n}{n}.$$

Check that $\log[q^\flat]$ converges in B_{dR}^+.

(vi) Let $u = \log[q^\flat]$. Show that $g(u) = u + a(g)t$.

(vii) Show that $V_p(E_q)$ is de Rham.
(Hint: Use t and u to modify the basis $e \otimes 1$, $f \otimes 1$ of $V_p(E_q) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ into a G_K-invariant one.)

Problem 20. Let n, m be two positive integers and $n \neq m$. Let V be any extension

$$0 \to \mathbb{Q}_p(n) \to V \to \mathbb{Q}_p(m) \to 0$$

in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Show that

(i) V is Hodge-Tate.

(ii) V is de Rham if $n > m$.
(On the other hand, every non-trivial extension

$$0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p(1) \to 0$$

is not de Rham. But this is difficult to prove.)

Problem 21. In this problem, we prove that the functor $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Fil}_K$ is not full.

(i) For $V, W \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$, show that $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{dR}}(W)$ are isomorphic in Fil_K if and only if

$$\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V)) = \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(W))$$

for all i. (i.e., they have the same Hodge-Tate weights and Hodge-Tate numbers.)

(ii) Show that there exists a non-split extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ in $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$.

(iii) Conclude that D_{dR} is not full.

5. Crystalline period ring B_{crys}

Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. Let $K_0 = W(k)[\frac{1}{p}]$. We will construct period ring B_{crys} equipped with both a filtration and a φ-action.

Recall that $\xi = [p^\flat] - p$. Consider

$$A^0_{\mathrm{crys}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})\Big[\frac{\xi^m}{m!}\Big]_{m \geq 1} \subset W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}].$$

This is a G_K-stable $W(\mathcal{O}_{\mathbb{C}_p^\flat})$-subalgebra generated by "divided-powers". Define

$$A_{\mathrm{crys}} = \varprojlim_n A^0_{\mathrm{crys}}/p^n A^0_{\mathrm{crys}}$$

with p-adic topology. We can fill in the top arrow in the commutative diagram

$$\begin{array}{ccc} A_{\mathrm{crys}} & \longrightarrow & B^+_{\mathrm{dR}} \\ \uparrow & & \uparrow \\ A^0_{\mathrm{crys}} & \longrightarrow & W(\mathcal{O}_{\mathbb{C}_p^\flat})[\frac{1}{p}] \end{array}$$

and then identify A_{crys} with a subring of B^+_{dR}. More precisely,

$$A_{\mathrm{crys}} = \Big\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \,\Big|\, a_n \in W(\mathcal{O}_{\mathbb{C}_p^\flat}),\ a_n \to 0 \ \ p\text{-adically} \Big\}$$

Define $B^+_{\mathrm{crys}} = A_{\mathrm{crys}}[\frac{1}{p}] \subset B^+_{\mathrm{dR}}$. Since $t \in A_{\mathrm{crys}}$, we can define

$$B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}] \subset B^+_{\mathrm{dR}}[\frac{1}{t}] \subset B_{\mathrm{dR}}$$

equipped with subspace topology from the new topology on B_{dR} introduced in Problem 11. Moreover, B_{crys} admits a natural action of G_K. There is a G_K-equivariant injection $K \otimes_{K_0} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$. Consequently, $B^{G_K}_{\mathrm{crys}} = K_0$.

The filtrations on B_{crys} are the ones inherited from B_{dR}. Namely, $\mathrm{Fil}^i B_{\mathrm{crys}} = \mathrm{Fil}^i B_{\mathrm{dR}} \cap B_{\mathrm{crys}}$. Unlike B_{dR}, the φ-action extends to A^0_{crys}, and hence on A_{crys}, B^+_{crys}, B_{crys}. (One can verify that $\varphi(t) = pt$.) However, the filtrations are not φ-stable.

Theorem 5.1. *$\varphi : A_{\mathrm{crys}} \to A_{\mathrm{crys}}$ is injective.*

Theorem 5.2. *We have "fundamental exact sequences"*

$$0 \to \mathbb{Q}_p \to B^{\varphi=1}_{\mathrm{crys}} \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0$$

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^0 B_{\mathrm{crys}} \xrightarrow{\varphi-1} B_{\mathrm{crys}} \to 0$$

The proof of these two fundamental results are difficult.

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Problem 22.

(i) Check that $t \in A_{\mathrm{crys}}$ and $t^{p-1} \in pA_{\mathrm{crys}}$. Consequently, $\frac{t^p}{p!} \in A_{\mathrm{crys}}$.

(ii) Show that for any $a \in \ker(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}_p})$, we have $\frac{a^m}{m!} \in A_{\mathrm{crys}}$, $\forall m \geq 1$.

Problem 23. Consider the G_K-equivariant injection $K \otimes_{K_0} B_{\mathrm{crys}} \hookrightarrow B_{\mathrm{dR}}$. Give left hand side the subspace filtration. Show that the induced map on the graded algebras is an isomorphism.

Problem 24. Check that A^0_{crys} is φ-stable.

Problem 25.

(i) Show that $B^+_{\mathrm{crys}} \subset \mathrm{Fil}^0 B_{\mathrm{crys}}$.

(ii) In this exercise, we show $B^+_{\mathrm{crys}} \neq \mathrm{Fil}^0 B_{\mathrm{crys}}$. Consider

$$\alpha = \frac{[\varepsilon^{1/p}] - 1}{[\varepsilon^{1/p^2}] - 1}.$$

Show that $\alpha \in B_{\mathrm{crys}}$, $\frac{1}{\alpha} \in B_{\mathrm{crys}} \cap B^+_{\mathrm{dR}}$, but $\frac{1}{\varphi(\alpha)} \notin B^+_{\mathrm{dR}}$. (This implies $\frac{1}{\alpha} \in \mathrm{Fil}^0 B_{\mathrm{crys}} - B^+_{\mathrm{crys}}$.)

Problem 26. Show that φ on B_{crys} does not preserve filtrations.

Problem 27. As pointed out in [Col], the topology on B_{crys} is unpleasant. In particular, the subspace topology inherited from B_{crys} is different from the one on B^+_{crys}. Let $\omega = ([\varepsilon] - 1)/([\varepsilon^{1/p}] - 1)$ and consider

$$x_n = \frac{\omega^{p^n}}{(p^n - 1)!}$$

(i) Show that $(x_n)_{n\geq 0}$ does not converge to 0 in B^+_{crys}.

(ii) Show that $(\omega x_n)_{n\geq 0}$ does converge to 0 in B^+_{crys} and hence $(x_n)_{n\geq 0}$ converges to 0 in B_{crys}.

Problem 28. A remedy to the topology issue in Problem 27 is to introduce $B_{\max}$. Define

$$A_{\max} = \Big\{ \sum_{n=0}^{\infty} a_n \frac{\omega^n}{p^n} \mid a_n \in W(\mathcal{O}_{\mathbb{C}_p^\flat}), a_n \to 0 \ \ p\text{-adically} \Big\}$$

and let $B^+_{\max} = A_{\max}[\frac{1}{p}] \subset B^+_{\mathrm{dR}}$, $B_{\max} = B^+_{\max}[\frac{1}{t}] \subset B_{\mathrm{dR}}$. Similar to B_{crys}, the ring $B_{\max}$ is equipped with G_K-action, φ-action, and filtration.

(i) Show that $B_{\max}$ does not have the issue in Problem 27.
(ii) Show that $B_{\max}^{G_K} = K_0$.
(iii) Show that $A_{\max}^{\varphi=1} = \mathbb{Z}_p$. Hence $(B^+_{\max})^{\varphi=1} = \mathbb{Q}_p$.
(iv) Show that $\varphi(B_{\max}) \subset B_{\mathrm{crys}} \subset B_{\max}$.
(v) (Hard!) Prove the analogue of Theorem 5.2: The following sequences are exact

$$0 \to \mathbb{Q}_p \to B_{\max}^{\varphi=1} \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0$$
$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^0 B_{\max} \xrightarrow{\varphi-1} B_{\max} \to 0$$

6. Crystalline representations

Let MF_K^{φ} denote the category of triples $(D, \varphi_D, \mathrm{Fil}^\bullet)$ where

- D is a K_0-vector space
- $\varphi_D : D \to D$ is a bijective φ-semilinear endomorphism
- $\mathrm{Fil}^\bullet$ is a filtration on $D_K = D \otimes_{K_0} K$ such that $(D_K, \mathrm{Fil}^\bullet)$ is an object in Fil_K.

Objects in MF_K^{φ} are called *filtered φ-modules.*

Consider functor

$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi}$$
$$V \mapsto \big(V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\big)^{G_K}$$

It is always true that $\dim_{K_0} D_{\mathrm{crys}}(V) \leq \dim_{\mathbb{Q}_p} V$. We say V is *crystalline* if $\dim_{K_0} D_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V$. The category of crystalline representations is denoted by $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_K)$.

Theorem 6.1. (i) *The functor*

$$D_{\mathrm{crys}} : \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi}$$

is exact and fully faithful. Moreover, it preserves direct sums, tensor products, subobjects, quotients, and duals.

(ii) *If V is crystalline, the natural map*

$$\alpha_{\mathrm{crys},V} : D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\mathrm{crys}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$$

is an isomorphism of filtered φ-modules.

(iii) *If V is crystalline, we can recover V from $D_{\mathrm{crys}}(V)$ by*

$$V = \mathrm{Fil}^0(D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\mathrm{crys}})^{\varphi=1}.$$

Theorem 6.2. *Crystalline representations are de Rham.*

Source of crystalline representations: those G_K-representations arising from p-adic étale cohomologies of proper smooth varieties over K with *good reduction* are crystalline.

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Problem 29. Describe $D_{\mathrm{crys}}(\mathbb{Q}_p(n))$ explicitly.

Problem 30. Let $\eta : G_K \to \mathbb{Q}_p^\times$ be a continuous character. Show that $\mathbb{Q}_p(\eta)$ is crystalline if and only if there exists $n \in \mathbb{Z}$ such that $\chi^n \eta$ is an unramified character.

Problem 31. Let D be a finite dimensional K_0-vector space and let $\varphi_D : D \to D$ be an injective φ-semilinear morphism. Prove that φ_D is automatically bijective.

Problem 32. Similar to D_{dR} and D_{crys}, we can consider

$$D_{\max} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi}$$
$$V \mapsto (V \otimes_{\mathbb{Q}_p} B_{\max})^{G_K}$$

It is always true that $\dim_{K_0} D_{\max}(V) \leq \dim_{\mathbb{Q}_p} V$. We say V is $B_{\max}$-*admissible* if $\dim_{K_0} D_{\max}(V) = \dim_{\mathbb{Q}_p} V$.

Prove that V is $B_{\max}$-admissible if and only if it is crystalline.

Problem 33. Let $V_p(E_q)$ be the representation studied in Problem 19.

(i) Is $V_p(E_q)$ crystalline?
(ii) Describe $D_{\mathrm{crys}}(V_p(E_q))$ explicitly.

Problem 34.(Hard!)

(i) Can you find an extension
$$0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$$
so that V is de Rham but not crystalline?
(ii) Show that any extension
$$0 \to \mathbb{Q}_p(n) \to V \to \mathbb{Q}_p \to 0$$
for $n \geq 2$ must be crystalline.

7. Period sheaves

This section dedicates to our first attempt on "relative period rings". We define $\mathbb{B}_{\mathrm{dR}}$ as a sheaf on the *pro-étale site* of adic spaces.

Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. The objects in the *pro-étale site* $X_{\mathrm{pro\acute{e}t}}$ are inverse systems

$$\varprojlim_{i \in I} U_i \to X$$

where $U_i \in X_{\mathrm{\acute{e}t}}$ and the transition maps $U_j \to U_i$ are finite étale surjective. The coverings are the topological ones. For details, the readers are referred to [Sch2]. One important property is that affinoid perfectoid objects in $X_{\mathrm{pro\acute{e}t}}$ form a basis for the pro-étale topology. Let $\mathcal{B}$ denote the collection of such objects. To define a sheaf, we only need to define a presheaf on $\mathcal{B}$.

To this end, we first define the period rings on affinoid perfectoids. Let (L, L^+) be a perfectoid field of characteristic 0. For any perfectoid affinoid (L, L^+)-algebra (R, R^+), we define

$$\mathbb{A}_{\inf}(R, R^+) := W(R^{\flat +})$$
$$\mathbb{B}_{\inf}(R, R^+) := \mathbb{A}_{\inf}(R, R^+)[\frac{1}{p}]$$
$$\mathbb{B}^+_{\mathrm{dR}}(R, R^+) := \varprojlim_n \mathbb{B}_{\inf}(R, R^+)/(\ker\theta)^n$$

where $\theta : \mathbb{A}_{\inf}(R, R^+) = W(R^{\flat +}) \to R^+$ is defined in the same way as in Section 3. Notice that ξ is a generator of θ (see Problem 35). We define $\mathbb{B}_{\mathrm{dR}}(R, R^+) = \mathbb{B}^+_{\mathrm{dR}}(R, R^+)[\frac{1}{\xi}]$. The filtration on $\mathbb{B}_{\mathrm{dR}}$ is given by $\mathrm{Fil}^i\,\mathbb{B}_{\mathrm{dR}} = \xi^i\mathbb{B}^+_{\mathrm{dR}}$, $i \in \mathbb{Z}$.

Back to the pro-étale site. We define a presheaf $\mathcal{F}_{\mathbb{A}_{\inf}}$ (resp., $\mathcal{F}_{\mathbb{B}_{\inf}}$, $\mathcal{F}_{\mathbb{B}^+_{\mathrm{dR}}}$, $\mathcal{F}_{\mathbb{B}_{\mathrm{dR}}}$) on $\mathcal{B}$ by sending $U = \mathrm{Spa}(R, R^+)$ to $\mathbb{A}_{\inf}(R, R^+)$ (resp., $\mathbb{B}_{\inf}(R, R^+)$, $\mathbb{B}^+_{\mathrm{dR}}(R, R^+)$, $\mathbb{B}_{\mathrm{dR}}(R, R^+)$). Finally, define $\mathbb{A}_{\inf}$ (resp., $\mathbb{B}_{\inf}$, $\mathbb{B}^+_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{dR}}$) to be the coresponding sheafifications.

These period sheaves played a central role in proving a de Rham comparison for rigid analytic varieties [Sch2]. Crystalline analogues are studied in [BMS], [TT]. Following the same spirit, sheaf versions of Robba rings and (φ, Γ)-modules are studied in [KL1], [KL2].

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Problem 35. For any perfectoid affinoid (L, L^+)-algebra (R, R^+), show that the kernel of $\theta : \mathbb{A}_{\inf}(R, R^+) \to R^+$ is a principle ideal generated by some $\xi \in \mathbb{A}_{\inf}(L, L^+)$.

Problem 36.

(i) Show that the presheaf $\mathcal{F}_{\mathbb{A}_{\inf}}$ on $\mathcal{B}$ satisfies sheaf properties. In particular, $\mathbb{A}_{\inf}(U) = \mathbb{A}_{\inf}(R, R^+)$, and $\mathbb{A}_{\inf} = W(\widehat{\mathcal{O}}^+_{X^\flat_{\mathrm{pro\acute{e}t}}})$.

(ii) Show that $H^i(U, \mathbb{A}_{\inf})$ is almost zero for all $i > 0$.

(iii) Show that $H^i(U, \mathbb{B}^+_{\mathrm{dR}}) = 0$ for all $i > 0$. (Hint: $[p^\flat]$ is invertible in $\mathbb{B}^+_{\mathrm{dR}}$.)

Problem 37.

(i) Construct period sheaves $\mathbb{A}^0_{\mathrm{crys}}$, $\mathbb{A}_{\mathrm{crys}}$, $\mathbb{B}^+_{\mathrm{crys}}$, $\mathbb{B}_{\mathrm{crys}}$, $\mathbb{B}^+_{\max}$, $\mathbb{B}_{\max}$ on $X_{\mathrm{pro\acute{e}t}}$ in the same way. Repeat Problem 36(i).

(ii) Show that $H^i(U, \mathbb{A}^0_{\mathrm{crys}})$ and $H^i(U, \mathbb{A}_{\mathrm{crys}})$ are almost zero for all $i > 0$.

(iii) Show that $H^i(U, \mathbb{B}^+_{\mathrm{crys}}) = 0$ for all $i > 0$.

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