ARIZONA WINTER SCHOOL 2017 OUTLINE: SHIMURA VARIETIES

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1. Course outline

The topic of this course is perfectoid Shimura varieties. The goal will be to explain how the theory of perfectoid spaces and the geometry of the Hodge-Tate period morphism enter into Scholze's breakthrough construction of Galois representations associated to torsion classes for GL_n. I plan to illustrate everything using modular curves, since even in this setting the underlying geometry is quite rich.

Specific aims for the course include:

(1) Providing context and motivation and giving examples of Shimura varieties (primarily modular curves) but also locally symmetric spaces which aren't Shimura varieties (Bianchi manifolds).
(2) Explaining why modular curves with infinite level at p are perfectoid, using the beautiful theory of the canonical subgroup.
(3) Introducing the Hodge-Tate period morphism, which maps a modular curve with infinite level at p to a much simpler geometric object, namely the flag variety $\mathbb{P}^1$ (considered as an adic space).
(4) Discussing the geometry of the period morphism, including the Newton stratification into the ordinary and the supersingular locus and what its fibers look like.
(5) Explaining a new method for constructing congruences and lifting mod p systems of Hecke eigenvalues to characteristic 0 via "fake-Hasse invariants" pulled back from the flag variety.

While I will mention Borel-Serre compactifications, higher-dimensional Shimura varieties and ingredients from p-adic Hodge theory, during the lecture series I will focus on the p-adic geometry of modular curves. These other topics will be discussed more extensively in the lecture notes.

2. Rough project description

For $\Gamma \subset GL_n(\mathbb{Q})$, the locally symmetric space X_Γ is a higher-dimensional analogue of a modular curve, except it doesn't have an algebraic structure for $n > 2$ (so it is not a Shimura variety). In general, the space X_Γ only has the structure of a real manifold. Still, its singular (or Betti) cohomology $H^*(X_\Gamma, \mathbb{C})$ can be computed in terms of higher-dimensional analogues of modular forms, which are at the center of the Langlands program. A question that has become more and more important in recent years is: what role does the Betti cohomology $H^*(X_\Gamma, \mathbb{F}_p)$ play and what can we prove about it?

One reason for studying $H^*(X_\Gamma, \mathbb{F}_p)$ comes from trying to prove the modularity of Galois representations. A recent insight of Calegari and Geraghty [2] is that understanding torsion as well as characteristic zero cohomology is crucial for proving modularity lifting theorems beyond the setting where the so-called Taylor-Wiles method applies. One input that Calegari and Geraghty require is the existence of Galois representations associated to torsion classes in the cohomology of locally symmetric spaces for GL_n, as realized by [7].

There is a subtlety, however: for applications to modularity, one needs to have Galois representations with coefficients in the Hecke algebra $\mathbb{T}$ acting on the cohomology of the locally symmetric space, not just Galois representations for individual $\bar{\mathbb{F}}_p$-systems of Hecke eigenvalues. This has been realized in [7] only up to a nilpotent ideal $I \subset \mathbb{T}$ of bounded, but possibly large nilpotence degree. Newton and Thorne refined this construction to get an ideal $I \subset \mathbb{T}$ such that $I^4 = 0$ [5]; after their work, the only remaining obstruction essentially comes from the excision long-exact sequence associated to the Borel-Serre compactification. Roughly, a class in the cohomology of the boundary could come from either compactly supported or usual cohomology of the interior and this ambiguity is the source of trouble.

The following project idea was suggested by Peter Scholze. The rough goal is the following:

(1) Prove that the compactly-supported cohomology of an appropriate Shimura variety (for the groups Sp_{2n} or $U(n,n)$) at level $\Gamma_0(p^\infty)$ (or perhaps $\Gamma_1(p^\infty)$) vanishes above the middle degree.
(2) Refine the arguments of [5] to construct the desired Galois representation (or determinant) by relating the locally symmetric space for GL_n to the cohomology of the corresponding Shimura variety at level $\Gamma_0(p^\infty)$ or $\Gamma_1(p^\infty)$.

The first part is a statement about a Shimura variety and so could be approachable with the tools developed by Scholze (in particular the theory of perfectoid spaces and the Hodge-Tate period morphism). At the same time, this could be a pretty delicate question in general.

We will start the week by discussing the first part in the case of modular curves. In this setting, the key idea to prove (1) is to exploit the fact that the anticanonical tower is already perfectoid at level $\Gamma_0(p^\infty)$, while the canonical tower is affinoid. Both of these extremes should give the desired bounds in this case.

If we are successful in the case of modular curves, the next step will be to understand subsets of higher-dimensional Shimura varieties with mixed behavior - not perfectoid, not affinoid, but somewhere in between. This part of the project is more speculative, but there should be a lot of nice geometry to explore.

Finally, if we are succesful on the side of the Shimura variety, we will move to thinking about the second step. This should involve a detailed study of the boundary of Borel-Serre compactifications.

3. Reading list

In addition to the detailed lecture notes, which should cover all the background topics needed for the course and for the project, I recommend the following sources for learning about perfectoid Shimura varieties and related topics:

- Scholze's paper on the construction of Galois representations associated to torsion classes which occur in the cohomology of locally symmetric spaces for GL_n [7]. This could be combined with the survey [6] for foundational results, especially for ingredients from p-adic Hodge theory. One should focus on Chapters 3 and 4 of [7] for following the lecture series.

- The joint paper [3] which explores the Hodge-Tate period morphism further. One should focus on Chapter 4 and specialize those results to modular curves.
- The survey [8], which also provides a lot of number-theoretic context and motivation.
- The survey [4], which is written at a more advanced level than [8] but highlights key aspects of the results in [7].

For the students in the project group, the role that the Borel-Serre compactification and its boundary play will be important as well as the geometry of perfectoid Shimura varieties. I recommend the following additional sources:

- Secton 5 of [7].
- The paper [1] for learning about completed cohomology.
- The paper [5] which improves the bound on the degree of nilpotence of I by working with derived variants of Hecke algebras. It also gives a different way of thinking about completed cohomology and describes the boundary of the Borel-Serre compactification carefully.

References

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[2] F. Calegari and D. Geraghty, *Modularity lifting beyond the Taylor-Wiles mehod*, https://www2.bc.edu/david-geraghty/files/merge.pdf.

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[4] S. Morel, *Construction de reprsentations galoisiennes de torsion*, Astrisque **380** (2016), 449-473.

[5] J. Newton and J. Thorne, *Torsion Galois representations over CM fields and Hecke algebras in the derived category*, arXiv:1511.04913.

[6] P. Scholze, *Perfectoid spaces: A survey*, Current Developments in Mathematics (2012).

[7] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Annals of Mathematics **182** (2015), no. 3, 945–1066.

[8] J. Weinstein, *Reciprocity laws and Galois representations: recent breakthroughs*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 1, 139.

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