§1. Recall:

- X_Γ, $\Gamma \subset SL_2(\mathbb{Z})$
  modular curve

(more generally, X_Γ could be a Shimura variety of Hodge type, e.g. GSp_{2g}, $g \in \mathbb{Z}_{\geq 1}$. $U(2,2)$)

- using Borel-Serre compactificati[on] we are reduced to understand[ing]

$$H^*_c(X_\Gamma, \mathbb{Z}/p^N\mathbb{Z})$$

- can also use toroidal compactificat[ions]
  Harris. Lan-Taylor-Thorne, Boxer

Thm*: Every system of Hecke eigenvals occurring in $H^*_c(X_\Gamma, \mathbb{Z}/p^n\mathbb{Z})$ lifts to char 0.

Rk: 1) Galois representations are constructed by congruences, not directly from étale cohomology

2). weight may change
level at p may change

Goal: understand pf of Thm*. Strategy:

1) use perfectoid modular curves

2) use Hodge-Tate period morphism

3) produce congruences using "fake" Hasse invariants.

§2. Analogy: we want a p-adic analogue of the following complex picture:

$$
\begin{tikzcd}
 & \mathcal{H}^2 \arrow[dl] \arrow[dr, hook, "\pi_{HdR}"] & \\
X_\Gamma(\mathbb{C}) & & \mathbb{P}^1(\mathbb{C})
\end{tikzcd}
$$

Borel embedding
holomorphic

flag variety parametrizing lines in $\mathbb{C}^2$

· the natural $GL_2(\mathbb{C})$-action on $\mathbb{P}^1(\mathbb{C})$ restricts to action of $SL_2(\mathbb{R})$ on $\mathcal{H}$ via Möbius transformations

· the picture has the following moduli interpretation

$$\begin{tikzcd}
 & \mathcal{H}^2 \left(\begin{array}{l} E/\mathbb{C} \text{ elliptic curve} \\ \alpha : H_1(E(\mathbb{C}), \mathbb{Z}) \\ \quad\quad \cong \\ \quad\quad \mathbb{Z}^2 \end{array} \right) \arrow[dl] \arrow[dr] & \\
X_\Gamma(\mathbb{C}) & & \mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

$\left(\begin{array}{l} E/\mathbb{C}, \Gamma\text{-level} \\ \text{structure} \end{array} \right)$

Hodge (-de Rham) filtration on E

$$\mathbb{C}^2 \underset{\alpha}{\simeq} H_1(E(\mathbb{C}), \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$$
$$\Big\downarrow\!\!\downarrow$$
$$\operatorname{Lie} E$$

<u>Rmk</u>: the above is used for defining "automorphic vector bundl

§3. Statement in p-adic case

- Let $G = GL_2/\mathbb{Q}$ (or GSp_{2g}, $g \in \mathbb{Z}_{\geq 1}$)

- For $m \in \mathbb{Z}_{\geq 0}$, define congruence subgps

$$\Gamma(p^m) = \left\{ \gamma \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; \begin{array}{l} \gamma = (\gamma^p, \gamma_p) \\ \gamma_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod p^m \\ \gamma^p \in \Gamma^p \subset GL_2(\hat{\mathbb{Z}}^p) \end{array} \right\}$$

Γ^p ← tame level

$\Gamma := \Gamma(p^m)$ for $m = 0$.

- Let $\mathbb{Q}_p^{cycl} := \mathbb{Q}_p(\mu_{p^\infty})^\wedge$ perfectoid field $\supset \mathbb{Z}_p^{cycl}$ integers

$X^*_{\Gamma(p^m)}$ modular curve of level $\Gamma(p^m)$, compactified

<u>Thm</u>: 1). $\exists$ perfectoid space $X^*_{\Gamma(p^\infty)}$ over $\mathbb{Q}_p^{cycl}$

s.t. $X^*_{\Gamma(p^\infty)} \sim \varprojlim_m X^*_{\Gamma(p^m)}$

adic space corresponding to $X^*_{\Gamma(p^m)}$ over $\mathbb{Q}_p^{cycl}$

- on topological spaces

$$|X^*_{\Gamma(p^\infty)}| = \varprojlim_m |X^*_{\Gamma(p^m)}|$$

homeomorphism

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• on structure sheaves: $\exists$ open cover of $\mathcal{X}^*_{\Gamma(p^\infty)}$ by affinoids $\mathrm{Spa}(A, A^+)$ s.t.

$$\varinjlim A_m \longrightarrow A$$

$\mathrm{Spa}(A_m, A_m^+) \subset \mathcal{X}^*_{\Gamma(p^m)}$ has dense image

map $\mathrm{Spa}(A, A^+) \to \mathcal{X}^*_{\Gamma(p^m)}$ factors through $\mathrm{Spa}(A_m, A_m^+)$

2). $\exists$ map of adic spaces

$$\pi_{HT}: \mathcal{X}^*_{\Gamma(p^\infty)} \longrightarrow \mathbb{P}^{1,ad}$$

adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}$

which can be described on points as follows:

Let C = complete, alg closed extension of $\overline{\mathbb{Q}}_p$, $C = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$

$\mathcal{X}^*$ (E, Γ-level str. : $\alpha : T_pE \simeq \mathbb{Z}_p^2$)

$\downarrow$

$\mathcal{X}^*_{\Gamma(p)}$ E/C, Γ-level str. $\alpha : E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$

$\downarrow$

$\mathcal{X}^*_{\Gamma}$

$\hookrightarrow$ elliptic curve E/C + Γ-level structure

$$\begin{tikzcd}[column sep=small] (E,\alpha)/C \arrow[rrdd, mapsto] & \mathcal{X}^*_{\Gamma(p^\infty)} \circlearrowleft GL_2(\mathbb{Q}_p) \arrow[d] \arrow[rd, "\pi_{HT}"] & \\ (E/C) \in \mathcal{X}^*_{\Gamma} & & \mathbb{P}^{1,ad} \circlearrowleft GL_2(\mathbb{Q}_p) \\ & & (\mathrm{Lie}\,E) \otimes_C C(1) \subset T_pE \otimes_{\mathbb{Z}_p} C \end{tikzcd}$$

$$T_pE \otimes_{\mathbb{Z}_p} C \underset{\alpha}{\cong} C^2$$

Hodge-Tate filtration on E

recall:

$$0 \to (\mathrm{Lie}\,E) \otimes_C C(1) \to T_pE \otimes_{\mathbb{Z}_p} C$$

$$\to \mathrm{Lie}\,E^\vee \to 0$$

- HT filtration not split in relative setting, i.e. when working w. a family of elliptic curves.
- to define HT filtration in relative setting, need to work over affinoid perfectoid cover of $\mathcal{X}^*_\Gamma$
- 1 & 2 of Thm are deeply intertwined

§4. Where does perfectoid structure come from?

Example: Let

$$\cdots \to \mathfrak{X}_n \xrightarrow{f_n} \cdots \to \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0$$

be a tower of flat formal schemes / $\mathrm{Spf}\, \mathbb{Z}_p^{cycl}$.

Assume f_n mod p factors as

$$\left(\mathfrak{X}_n \otimes_{\mathbb{Z}_p^{cycl}} \mathbb{Z}_p^{cycl}/p\right) \xrightarrow{\text{rel Frob}}$$

$$\left(\mathfrak{X}_n \otimes_{\mathbb{Z}_p^{cycl}} \mathbb{Z}_p^{cycl}/p\right)^{(p)} \simeq$$

$$\xrightarrow{\sim} \left(\mathfrak{X}_{n-1} \otimes_{\mathbb{Z}_p^{cycl}} \mathbb{Z}_p^{cycl}/p\right)$$

Then $\mathfrak{X}_\infty := \varprojlim_n \mathfrak{X}_n$

(in cat of formal schemes / $\mathbb{Z}_p^{cycl}$

gives generic fibre $X_\infty = (\mathfrak{X}_\infty)_\eta$

which is a perfectoid space

over $\mathbb{Q}_p^{cycl}$ &

$$X_\infty \sim \varprojlim_n X_n$$

X_n = adic generic fibre

of $\mathfrak{X}_n$

E.g. $\mathfrak{X}_n = \mathrm{Spf}\, \mathbb{Z}_p^{cycl}\langle t\rangle$

$f_n : \mathfrak{X}_n \longrightarrow \mathfrak{X}_{n-1}$
$t^p \leftarrow\!\shortmid t$

gives

$$X_\infty = \mathrm{Spa}\left(\mathbb{Q}_p^{cycl}\langle t^{1/p^\infty}\rangle, \mathbb{Z}_p^{cycl}\langle t^{1/p^\infty}\rangle\right)$$

We'll show that part of $(X_{\Gamma_0(p^m)})_m$ has this behaviour

$$\Gamma_0(p^m) = \left\{ \gamma \text{ — — } \cdot \quad \gamma_p \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^m \quad \det \gamma_p \equiv 1 \bmod p^m \right\}$$