§1. Reciprocity laws

{ spectral data, e.g. systems of eigenvalues, arising from topology of highly symmetric manifolds } »automorphic side"

$\updownarrow$

{ arithmetic data, from counting solutions to diophantine equations } "Galois side"

A.S: locally symmetric spaces

G / F

G: conn. red group, e.g. GL_n, SL_n, Sp_{2n}

F: number field

2). $G = SL_2 \,/\, F = \mathbb{Q}(i)$ $\quad \mathbb{C} \cong \mathbb{R}^2$ ②

symmetric space $X = SL_2(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{R}) \big/ SU_2(\mathbb{C})$

$= SL_2(\mathbb{C}) / SU_2(\mathbb{C})$

$\cong$

$\mathcal{H}^3$ hyperbolic 3-space

$\Gamma \subseteq SL_2(\mathbb{Z}[i])$
congruence subgp

$\rightsquigarrow X_\Gamma = {}^{X}\!/_{\Gamma}$ locally symmetric space for $\mathrm{Res}_{F/\mathbb{Q}} SL_2$

arithmetic hyperbolic 3-manifold
(Bianchi manifold)

no direct connection to algebraic geometry!

G.S: varieties def by polynomial eq'ns w coeffs in F

Examples:

1). $G = SL_2 / \mathbb{Q}$

symmetric space for SL_2

$$X = SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$
$$\cong$$
$$\mathcal{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

natural complex structure

$\Gamma \subseteq SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) \curvearrowright X$

congruence subgp

$X_\Gamma = \Gamma \backslash X$ locally symmetric space for SL_2

Riemann surface

§2. The case of $SL_2/\mathbb{Q}$

Reciprocity law:

f modular form

$$f(z) = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2$$

$$= \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i z}$$

→ holomorphic fn on $\mathcal{H}^2$

- satisfying symmetries under $\Gamma = \Gamma_0(11)$

$$\left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 11 \right\}$$

including $f(z+1) = f(z)$

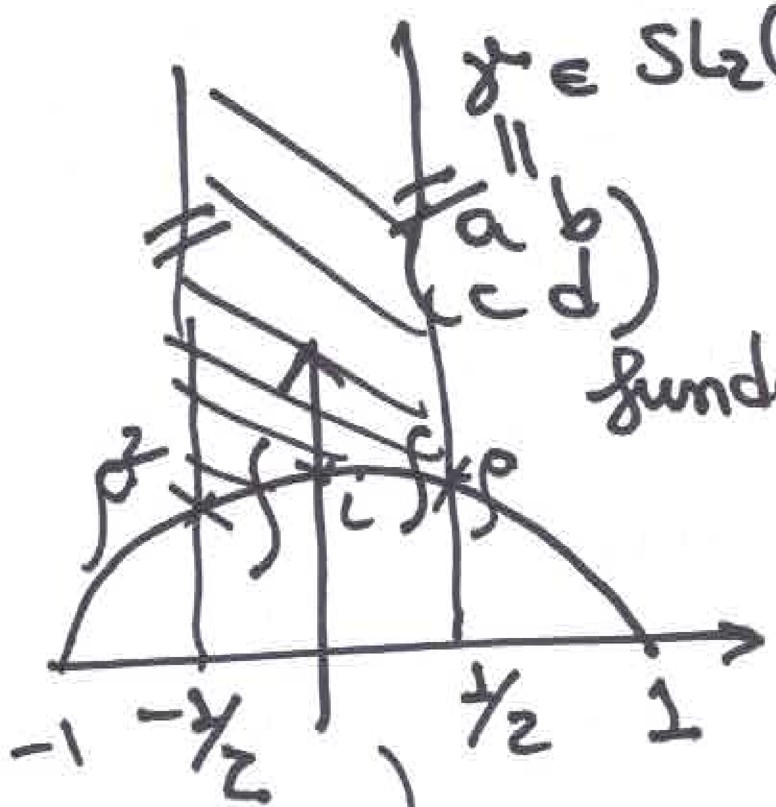
⇝ Fourier series

- satisfying a growth condition.

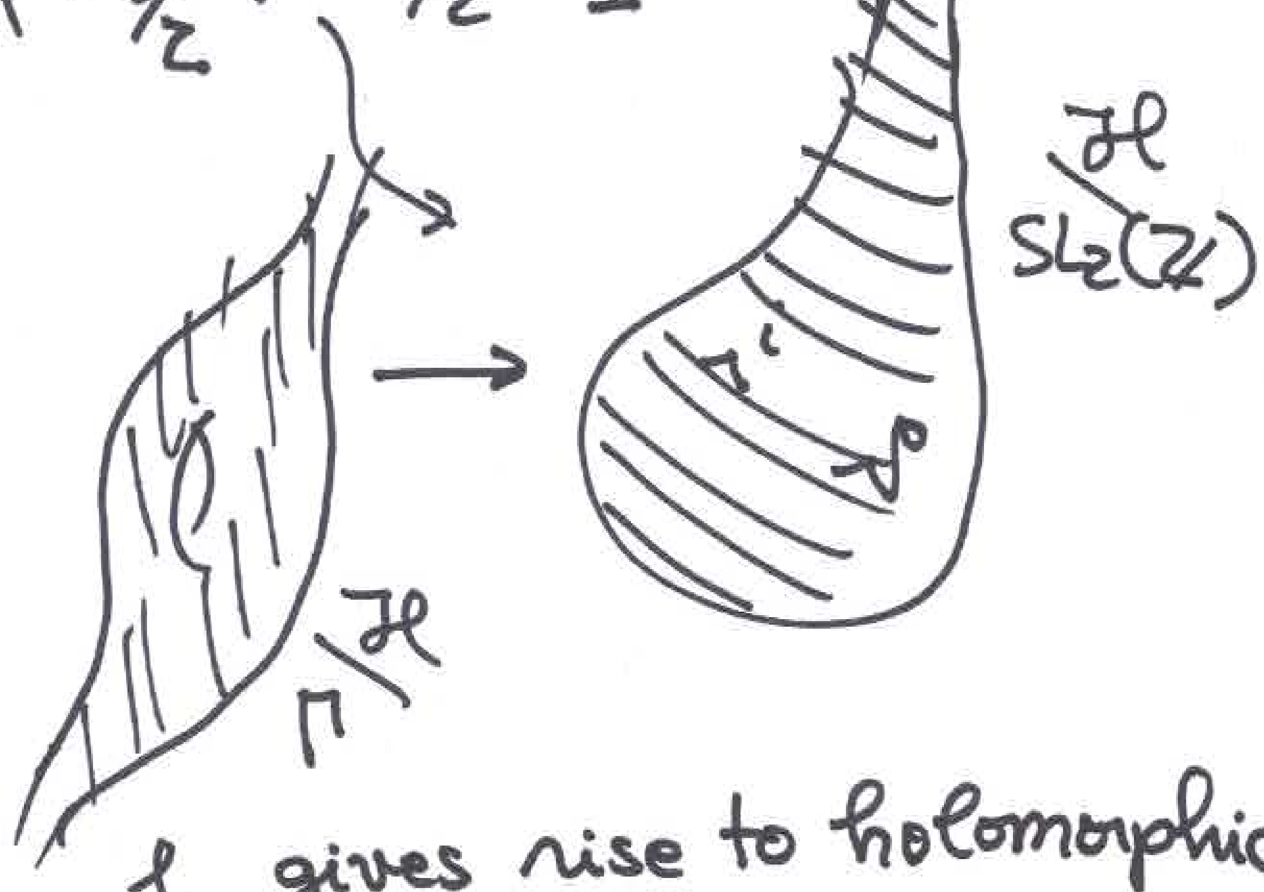
$\Gamma \subset SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) \curvearrowright \mathcal{H}^2$

$\gamma \in SL_2(\mathbb{Z}) \quad \gamma : z \longmapsto \dfrac{az+b}{cz+d}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



fundamental domain for $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}^2$



f gives rise to holomorphic differential ω_f on $\Gamma \backslash \mathcal{H}$

ℓ prime, $\ell \neq 11$, a_ℓ = eigenvalue of T_ℓ acting on f

$\rightsquigarrow$ $(a_\ell)_{\ell \text{ prime}, \neq 11}$ is a <u>system of Hecke eigenvalues</u>

this is the <u>spectral data</u>

$E/\mathbb{Q}$ elliptic curve

$y^2 + y = x^3 - x^2$

<u>arithmetic data</u> $(\#E(\mathbb{F}_\ell))_{\ell \text{ prime} \neq 11}$

Explicit reciprocity law:

$$\boxed{a_\ell = \ell + 1 - \#E(\mathbb{F}_\ell)}$$

More sophisticated version of reciprocity: Galois representations

$$\begin{tikzcd} f \arrow[dr, squiggly] & E \arrow[d] & \\ & \rho_f \simeq \rho_E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[d] \\ & GL_2(\mathbb{Q}_p) \end{tikzcd}$$

- modularity of elliptic curves
- uses congruences mod p^n

1). ρ_E is given by $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action on $T_p E$

$$T_p E = \varprojlim_n E[p^n].$$

This is a special case of obtaining $G_{\mathbb{Q}}$-reps from étale cohomology of alg vars$/\mathbb{Q}$.

2). how to construct ρ_f ? ⑧

- $f \rightsquigarrow \omega_f$ hol diff on $\Gamma\backslash\mathcal{H}$

Hodge $\overset{*}{\rightsquigarrow}$ system of eigenvalues occurring in decomposition $H^1(\Gamma\backslash\mathcal{H}, \underset{\mathbb{Q}}{\mathbb{C}})$

$*$ - this is subtle because $\Gamma\backslash\mathcal{H}$ is non-compact

- <u>miracle</u> : $\exists$ alg curve
  $X_\Gamma / \mathbb{Q}$ modulon curve
  s.t. $\Gamma\backslash\mathcal{H} = X_\Gamma(\mathbb{C})$

why? $\mathcal{H}$ parametrizes complex structures on $\mathbb{R}^2$

$\cong$

$\mathcal{H}$odge structures ~~of~~ on $H^1(E(\mathbb{C}), \underset{\mathbb{R}^2}{\mathbb{R}})$

→ $\Gamma\backslash\mathcal{H}$ moduli of elliptic curves / $\mathbb{C}$ + extra structures

· moduli problem makes sense / $\mathbb{Q}$ & is representable by X_Γ.

<u>Upshot</u>: ρ_f is "cut out" from étale cohomology $H^1_{et}(X_\Gamma \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p)$

This roughly generalizes to other groups

$GSp_{2g}/\mathbb{Q}$, unitary groups

i.e. to groups that admit Shimura varieties

Works when X is a moduli of Hodge structures of abelian varieties

§3. The case of SL_2/F.

$$X_\Gamma = \Gamma \backslash \mathcal{H}^3, \quad \Gamma \subset SL_2(\mathbb{Z}[i])$$

· Still have: $H^*(X_\Gamma, \mathbb{C})$
can be related
to generalizations
of modular forms.

· Problems: 1). no direct
connection to
algebraic geometry
2). need to understand
$H^*(X_\Gamma, \mathbb{Z}/p^N\mathbb{Z})$
includes a lot of torsion.

Strategy: Can relate X_Γ (arithmetic hyperbolic 3-manifolds)

to $\tilde{X}_{\tilde{\Gamma}}$ ← a $U(2,2)$ Shimura variety

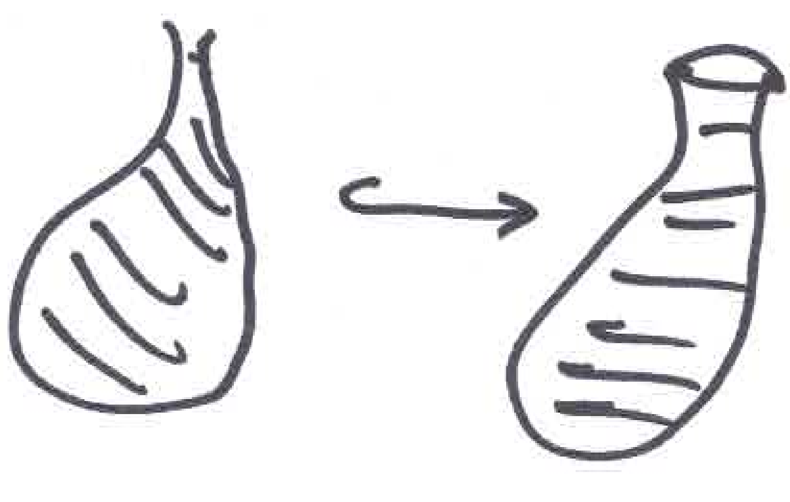
parametrizes AV's of dim 4.

via Borel-Serre compactification

$$\tilde{X}_{\tilde{\Gamma}} \hookrightarrow \tilde{X}_{\tilde{\Gamma}}^{BS}$$

homotopy equivalence

real manifold w corners.



Borel-Serre compactification of $_{\Gamma}\backslash\mathcal{H}$

excision long exact sequence

$$H^i_c(\tilde{X}_{\widehat{\Gamma}}, \mathbb{Z}/p^n\mathbb{Z}) \to H^i(\tilde{X}_{\widehat{\Gamma}}, \mathbb{Z}/p^n\mathbb{Z}$$

$$\to H^i(\partial\tilde{X}^{BS}_{\widehat{\Gamma}}, \mathbb{Z}/p^n\mathbb{Z})$$

$$\to H^{i+1}_c(\tilde{X}_{\widehat{\Gamma}}, \mathbb{Z}/p^n\mathbb{Z})$$

see X_Γ here

(arithmetic hyperbolic 3-manif

__Upshot__: enough to understand

$$H^*_c(\tilde{X}_{\widehat{\Gamma}}, \mathbb{Z}/p^n\mathbb{Z})$$

Shimura variety

$\mathrm{Res}_{F/\mathbb{Q}}\, GL_2$ is a Levi subgp in max'l parabolic of $\mathcal{U}(2,2)$