The Hodge-Tate decomposition via perfectoid spaces
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Contents

1. Lecture 1: Introduction

1.1 Statement and consequences of the Hodge-Tate decomposition

Fix a prime number p. The goal of this series is to explain the p-adic analog of the following classical result, which forms the starting point of Hodge theory.

Theorem 1.1.1 (Hodge decomposition)**.** *Let $X/\mathbf{C}$ be a smooth proper variety. Then there exists a natural isomorphism*

$$H^n(X^{an}, \mathbf{C}) \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbf{C}}).$$

Theorem 1.1.1 has many immediate consequences. For example, the "naturality" assertion above implies that the Hodge numbers are topological invariants in the following sense:

Corollary 1.1.2. *If $f : X \to Y$ is a map of smooth proper varieties that induces an isomorphism $H^n(Y^{an}, \mathbf{C}) \simeq H^n(X^{an}, \mathbf{C})$ for some $n \geq 0$, then one also has $H^i(Y, \Omega^j_{Y/\mathbf{C}}) \simeq H^i(X, \Omega^j_{X/\mathbf{C}})$ for each i, j with $i + j = n$.*

To move towards the p-adic analog, recall that the theory of étale cohomology provides an algebraic substitute for singular cohomology that works over any field k: the two roughly coincide when $k = \mathbf{C}$, but the former is constructed directly from algebraic geometry, and thus witnesses the action of algebraic symmetries, including those that might not be holomorphic when working over $k = \mathbf{C}$. As a concrete consequence, we have the following vaguely formulated statement:

Theorem 1.1.3 (Grothendieck, Artin, ...)**.** *Let $X/\mathbf{C}$ be an algebraic variety that is defined over $\mathbf{Q}$. Then the absolute Galois $G_{\mathbf{Q}}$ of $\mathbf{Q}$ acts canonically on $H^i(X^{an}, \mathbf{Z}/n)$ for any integer $n > 0$. Letting n vary through powers of a prime p, we obtain a continuous $G_{\mathbf{Q}}$-action on the $\mathbf{Z}_p$-module $H^i(X^{an}, \mathbf{Z}_p)$, and thus on the $\mathbf{Q}_p$-vector space $H^i(X^{an}, \mathbf{Q}_p)$.*

Some important examples of this action are:

Example 1.1.4 (Elliptic curves)**.** Let $X = E$ be an elliptic curve over $\mathbf{C}$ which is defined over $\mathbf{Q}$. Then

$$H^1(X^{an}, \mathbf{Z}/n) \simeq H_1(X^{an}, \mathbf{Z}/n)^\vee \simeq \mathrm{Hom}(\pi_1(E), \mathbf{Z}/n) \simeq E[n]^\vee$$

is the $\mathbf{Z}/n$-linear dual of the n-torsion of E. In this case, Theorem 1.1.3 reflects the fact that all n-torsion points on E are defined over $\overline{\mathbf{Q}} \subset \mathbf{C}$, and are permuted by the Galois group $G_{\mathbf{Q}}$ as E has $\mathbf{Q}$-coefficients. Passing to the inverse limit, this endows the p-adic Tate module $T_p(E) := \lim_n E[n]$ and its $\mathbf{Z}_p$-linear dual

$H^1(X^{an}, \mathbf{Z}_p)$ with canonical $G_{\mathbf{Q}}$-actions. As $T_p(E) \simeq \mathbf{Z}_p^2$ as a topological group, this discussion provides a continuous 2-dimensional representation $G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p)$. More generally, the same discussion applies to any abelian variety of dimension g to yield a continuous representation $G_{\mathbf{Q}} \to \mathrm{GL}_{2g}(\mathbf{Z}_p)$.

Example 1.1.5 (The torus and Tate twists)**.** Another important example is the case of $X = \mathbf{G}_m$. In this case, by the same reasoning above, we have $H^1(X^{an}, \mathbf{Z}/n) \simeq \mu_n^\vee$ (where $\mu_n \subset \overline{\mathbf{Q}}^*$ denotes the set of n-th roots of 1) and $H^1(X^{an}, \mathbf{Z}_p) \simeq (\lim_n \mu_n)^\vee =: \mathbf{Z}_p(1)^\vee =: \mathbf{Z}_p(-1)$. It is easy to see that $\mathbf{Z}_p(-1)$ is a rank 1 free module over $\mathbf{Z}_p$, so we can make sense of $\mathbf{Z}_p(j)$ for any integer j. Moreover, the resulting represenation $G_{\mathbf{Q}} \to \mathrm{GL}_1(\mathbf{Z}_p)$ is highly non-trivial by class field theory. In general, for a $\mathbf{Z}_p$-algebra R, we shall write $R(i) := R \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(i)$, and refer to this as the i-th Tate twist of R.

Example 1.1.6 (Projective line and abelian varieties)**.** Standard computations in algebraic topology are compatible with the Galois action from Theorem 1.1.3. Thus, for example, if $A/\mathbf{C}$ is an abelian variety of dimension g, then we know from topology that $H^*(A^{an}, \mathbf{Z}_p)$ is an exterior algebra on $H^1(A, \mathbf{Z}_p)$: we have $A \simeq (S^1)^{2g}$ as a topological space, so the claim follows by Künneth. It follows that the same description also applies in the world of $G_{\mathbf{Q}}$-modules. Likewise, via the Mayer-Vietoris sequence, we have a canonical isomorphism $H^2(\mathbf{P}^{1,an}, \mathbf{Z}_p) \simeq H^1(\mathbf{G}_m^{an}, \mathbf{Z}_p) \simeq \mathbf{Z}_p(-1)$ in the world of $G_{\mathbf{Q}}$-modules. More generally, if $X/\mathbf{C}$ is a smooth (or merely irreducible) projective variety of dimension d defined over $\mathbf{Q}$, then one can show $H^{2d}(X^{an}, \mathbf{Z}_p) \simeq \mathbf{Z}_p(-d)$ as a $G_{\mathbf{Q}}$-module.

From here on, we assume that the reader is familiar with the basics of étale cohomology theory[1]. Via Theorem 1.1.3 (and variants), this theory provides perhaps the most important examples of $G_{\mathbf{Q}}$-representations on p-adic vector spaces. To understand these objects, at a first approximation, one must understand the action of the local Galois groups (or decomposition groups) $D_\ell \subset G_{\mathbf{Q}}$ for a rational prime ℓ. When $\ell \neq p$, these actions can be understood[2] in terms of algebraic geometry over the finite field $\mathbf{F}_\ell$; in effect, due to the incompatibility of the ℓ-adic nature of D_ℓ with the p-adic topology, these actions are classified by the action of a single endomorphism (the Frobenius), and one has powerful tools coming from the solution of Weil conjectures at our disposal to analyze this endomorphism. However, if $\ell = p$, the resulting representations are much too rich to be understood in terms of a single endomorphism. Instead, these representations are best viewed as p-adic analogs of Hodge structures, explaining the name "p-adic Hodge theory" given to the study of these representations. Perhaps the first general result justifying this choice of name is the following, which gives the p-adic analog of the Hodge decomposition in Theorem 1.1.1 and forms the focus of this lecture series:

Theorem 1.1.7 (Hodge-Tate decomposition)**.** *Let $K/\mathbf{Q}_p$ be a finite extension, and let $\mathbf{C}_p$ be a completion of an algebraic closure $\overline{K}$ of K. Let X/K be a smooth proper variety. Then there exists a Galois equivariant decomposition*

$$H^n(X_{\overline{K},et}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbf{C}_p(-j), \tag{1.1}$$

[1] For a scheme X, a prime number p and a coefficient ring $\Lambda \in \{\mathbf{Z}/p, \mathbf{Z}/p^n, \mathbf{Z}_p, \mathbf{Q}_p\}$, we write $H^*(X_{et}, \Lambda)$ for the étale cohomology X with Λ-coefficients; we indulge here in the standard abuse of notation where, for $\Lambda \in \{\mathbf{Z}_p, \mathbf{Q}_p\}$, the groups $H^n(X_{et}, \Lambda)$ are not the cohomology groups of a sheaf on the étale site X_{et}, but rather are defined by an inverse limit procedure.

[2] We are implicitly assuming in this paragraph that the prime p is a prime of good reduction for the variety under consideration. If X has bad reduction at p, then the resulting representations of D_ℓ are much more subtle: already when $\ell \neq p$, there is an extremely interesting additional piece of structure, called the monodromy operator, that is still not completely understood.

where $\mathbf{C}_p(-j)$ denotes the $(-j)$-th Tate twist of $\mathbf{C}_p$. This isomorphism is functorial in X. In particular, it respects the natural graded algebra structures on either side as n varies.

We take a moment to unravel this statement. The object $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$ is the étale cohomology of $X_{\overline{K}} := X \otimes_K \overline{K}$, and hence admits a $G_K := \mathrm{Gal}(\overline{K}/K)$-action by transport of structure. The G_K-action on $\overline{K}$ is continuous, and hence extends to one on the completion $\mathbf{C}_p$. In particular, G_K acts on the left side of (1.1) via the tensor product action. On the right side, the only nontrivial G_K-action exists on Tate twists $\mathbf{C}_p(-j) := \mathbf{C}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)^{\otimes -j}$, where it is defined as the tensor product of G_K-actions on the two pieces. In particular, $\mathbf{C}_p(-j)$ is *not* a linear representation of G_K on a $\mathbf{C}_p$-vector space; instead, it is semilinear with respect to the standard G_K-action on $\mathbf{C}_p$.

To extract tangible consequences from Theorem 1.1.7, it is important to know that the Tate twists $\mathbf{C}_p(j)$ are distinct for different values of j. In fact, one has the much stronger statement that these Tate twists do not talk to each other for different values of j (see [Ta]):

Theorem 1.1.8 (Tate)**.** *Fix notation as in Theorem 1.1.7. Then, for $i \neq 0$, we have*

$$H^0(G_K, \mathbf{C}_p(i)) = H^1(G_K, \mathbf{C}_p(i)) = 0.$$

For $i = 0$, each of these groups is a copy of K. In particular, we have

$$\mathrm{Hom}_{G_K, \mathbf{C}_p}(\mathbf{C}_p(i), \mathbf{C}_p(j)) = 0$$

for $i \neq j$.

We now revisit the preceding examples.

Example 1.1.9. Consider $X := \mathbf{P}^1$ and $n = 2$. In this case, we have $H^2(X_{\overline{K},et}, \mathbf{Q}_p) \simeq \mathbf{Q}_p(-1)$ by Example 1.1.6 (see also Example 1.1.5). Using Theorem 1.1.8, we see that Theorem 1.1.7 captures the statement that $H^0(X, \Omega^2_{X/K}) = H^2(X, \mathcal{O}_X) = 0$, while $H^1(X, \Omega^1_{X/K})$ is 1-dimensional.

Example 1.1.10. Let $X = A$ be an abelian variety over K. By combining Example 1.1.4 and Theorem 1.1.7, we learn that

$$T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{C}_p \simeq \big(H^1(A, \mathcal{O}_A)^\vee \otimes_K \mathbf{C}_p\big) \oplus \big(H^0(A, \Omega^1_{A/K})^\vee \otimes_K \mathbf{C}_p(1)\big).$$

One can identify the right side in more classical terms:

$$H^0(A, \Omega^1_{A/K})^\vee \simeq \mathrm{Lie}(A) \quad \text{and} \quad H^1(A, \mathcal{O}_A) \simeq \mathrm{Lie}(A^\vee),$$

where $A^\vee$ is the dual of A. Thus, we can rewrite the above decomposition as

$$T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{C}_p \simeq \big(\mathrm{Lie}(A^\vee)^\vee \otimes_K \mathbf{C}_p\big) \oplus \big(\mathrm{Lie(A)} \otimes_K \mathbf{C}_p(1)\big).$$

As we shall see later, if A is merely defined over $\mathbf{C}_p$ instead of over a finite extension K as above, then we always have a short exact sequence

$$0 \to \mathrm{Lie}(A)(1) \to T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{C}_p \to \mathrm{Lie}(A^\vee)^\vee \to 0,$$

but this sequence may not split in a canonical way: there is no Galois action present when A is defined merely over $\mathbf{C}_p$, so one cannot invoke Theorem 1.1.8 to obtain a (necessarily unique!) splitting of the previous sequence.

In number theory, one of the main applications of these ideas is in understanding the Galois representations of G_K arising as $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$. For example, Theorem 1.1.7 implies these representations are Hodge-Tate, which forms the first in a series of increasingly stronger restrictions placed on the representations arising in this fashion from algebraic geometry; upgrading this structure, one can even give a completely "linear algebraic" description of these Galois representations (see Remark 1.2.4), which is very useful for computations.

Theorem 1.1.7 also has applications to purely geometric statements. For example, applying Theorem 1.1.8 leads to the following concrete consequence concerning the recovery of the algebro-geometric invariants $H^i(X, \Omega^j_{X/K})$ from the topological/arithmetic invariant $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$:

Corollary 1.1.11 (Recovery of Hodge numbers)**.** *With notation as in Theorem 1.1.7, we have*

$$H^i(X, \Omega^j_{X/K}) \simeq \Big(H^{i+j}(X_{\overline{K},et}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p(j)\Big)^{G_K}.$$

Proof. Set $n = i + j$. Tensoring both sides of (1.1) (and replacing j with k in that formula)

$$H^n_{et}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p(j) \simeq \bigoplus_{i+k=n} H^i(X, \Omega^k_{X/K}) \otimes_K \mathbf{C}_p(j-k).$$

Applying $(-)^{G_K}$ then gives the claim as $\mathbf{C}_p(j-k)^{G_K} = 0$ when $j \neq k$ by Theorem 1.1.8. □

In particular, Corollary 1.1.11 gives an analog of Corollary 1.1.2 in this setting. In fact, Corollary 1.1.11 is one of the key steps in Ito's alternative proof [It] of the following purely geometric result; the first proof of the latter gave birth to the theory of motivic integration [Ko, DL], and both proofs rely on Batyrev's [Ba] proving the analogous claim for Betti numbers via p-adic integration.

Theorem 1.1.12 (Kontsevich, Denef, Loeser, Ito)**.** *Let X and Y be smooth projective varieties over $\mathbf{C}$. Assume that both X and Y are Calabi-Yau (i.e., K_X and K_Y are trivial), and that X is birational to Y. Then*

$$\dim(H^i(X, \Omega^j_{X/\mathbf{C}})) = \dim(H^i(Y, \Omega^j_{Y/\mathbf{C}})).$$

for all i, j.

One may view Theorem 1.1.7 as relating the Galois representation on $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$ to the algebraic geometry of X. An obvious question that then arises, and one we have essentially skirted in the discussion so far, is whether one can understand the p-torsion in $H^n(X_{\overline{K},et}, \mathbf{Z}_p)$ in terms of the geometry of X; in particular, we may ask for a geometric description of $H^n(X_{\overline{K},et}, \mathbf{F}_p)$. This integral story is much less understood than the rational theory above. Nevertheless, one has the following recent result [BMS2], giving partial progress:

Theorem 1.1.13. *Fix notation as in Theorem 1.1.7. Assume that X extends to a proper smooth $\mathcal{O}_K$-scheme $\mathfrak{X}$. Write $\mathfrak{X}_k$ for the fiber of $\mathfrak{X}$ over the residue field k of K. Then we have*

$$\dim_{\mathbf{F}_p}(H^n(X_{\overline{K},et}, \mathbf{F}_p)) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}).$$

Moreover, there exist examples where the inequality is strict.

In other words, the mod-p cohomology of $X_{\overline{K}}$ is related to the geometry of $\mathfrak{X}_k$. We shall sketch a proof of Theorem 1.1.13 towards the end of the lecture series.

Outline of proof and lectures

The main goal of this lecture series is to explain a proof of Theorem 1.1.7; towards the end, we shall also sketch some ideas going into Theorem 1.1.13. Our plan is to prove Theorem 1.1.7 following the perfectoid approach of Scholze [Sc2], which itself is inspired by the work of Faltings [Fa1, Fa2, Fa3, Fa4]. In broad strokes, there are two main steps:

1. *Local study of Hodge cohomology via perfectoid spaces:* Construct a pro-étale cover $X_\infty \to X$ which is "infinitely ramified in characteristic p", and study the cohomology of X_∞. In fact, X_∞ shall be an example of a perfectoid space [Sc1], so the perfectoid theory gives a lot of control on the cohomology of X_∞. In particular, suitably interpreted, X_∞ carries no differential forms, so the full Hodge cohomology comes from the structure sheaf.

2. *Descent:* Descend the preceding understanding of the Hodge cohomology of X_∞ down to X. In this step, we shall see that the differential forms on X, which vanished after pullback to X_∞, reappear in the descent procedure.

In fact, to illustrate this process in practice, we work out explicitly the case of abelian varieties with good reduction in §2. The general case is then treated in §3, while the integral theory is surveyed in §4.

1.2 Complementary remarks

We end this section by some remarks of a historical nature, complementing the theory discussed above.

Remark 1.2.1. Theorem 1.1.7 was conjectured by Tate [Ta]. In the same paper, Tate also settled the case of abelian varieties (and, more generally, p-divisible groups) with good reduction; the case of general abelian varieties was then settled by Raynaud using the semistable reduction theory. The abelian variety case was revisited by Fontaine in [Fo1], who also provided a natural "differential" definition of the Tate twist. The general statement mentioned above was established by Faltings [Fa1], as a consequence of his machinery of almost étale extensions.

Remark 1.2.2. (Hodge-Tate decomposition for rigid spaces) In [Ta, §4, Remark], Tate wondered if Theorem 1.1.7 should be valid more generally for any proper smooth rigid-analytic[3] space. This question was answered affirmatively by Scholze [Sc2, Corollary 1.8]. In fact, Scholze proves the following more general assertion (see [Sc3, Theorem 3.20]):

Theorem 1.2.3 (Hodge-Tate filtration)**.** *Let C be a complete and algebraically closed nonarchimedean extension of $\mathbf{Q}_p$. Let X/C be a proper smooth rigid-analytic space. Then there exists an E_2-spectral sequence*

$$E_2^{i,j} : H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X_{et}, \mathbf{Q}_p) \otimes C.$$

[3] At first glance, this is very surprising: in complex geometry, the Hodge decomposition in Theorem 1.1.1 only applies to compact complex manifolds which are (not far from) Kähler, so one would also expect an analog of the Kähler condition in p-adic geometry. However, if one accepts that Kähler metrics are somewhat analogous to formal models (for example, the latter provides a well-behaved metric on the space of analytic functions), then the analogy with complex geometry is restored: as every rigid space admits a formal model by Raynaud [BL2, Theorem 4.1].

When X is defined over a discretely valued subfield of C (such as a finite extension of $\mathbf{Q}_p$), then this spectral sequence degenerates canonically due to Theorem 1.1.8 (which holds true for over any such field), leading to the Hodge-Tate decomposition for proper smooth rigid-analytic spaces, as inquired by Tate. It is this more general result that is most naturally accessible to perfectoid techniques, and thus forms the focus of this lecture series.

Remark 1.2.4 (p-adic comparison theorems)**.** The Hodge-Tate decomposition forms the first in a hierarchy of increasingly stronger statements (conjectured by Fontaine, and proven by various authors) describing the Galois representations of G_K on $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$ in terms of the geometry of X. For example, the p-adic de Rham comparison isomorphism, which is formulated in terms of a certain filtered G_K-equivariant $\overline{K}$-algebra B_{dR} constructed by Fontaine, asserts:

Theorem 1.2.5 (de Rham comparison)**.** *There exists a canonical isomorphism*

$$H^n(X_{\overline{K},et}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{dR} \simeq H^n_{dR}(X/K) \otimes_K B_{dR}.$$

This isomorphism respects the Galois action and filtrations.

Theorem 1.2.5, together with some knowledge of B_{dR}, allows one to recover the de Rham cohomology $H^n_{dR}(X/K)$ as a *filtered vector space* from the G_K-representation $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$. In fact, passage to the associated graded in Theorem 1.2.5 recovers Theorem 1.1.7, so one may view the de Rham comparison isomorphism as a non-trivial deformation of the Hodge-Tate decomposition. Continuing further, in the setting of good or semistable reduction, one can endow $H^n_{dR}(X/K)$ with some extra structure (namely, a Frobenius endomorphism, as well a monodromy endomorphism in the semistable case); the crystalline/semistable comparison theorems give an analogous comparison relating the G_K-representation $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$ with the de Rham cohomology $H^n_{dR}(X/K)$ equipped with the aforementioned additional structure. A major advantage of these latter theorems is that the recovery process works in both directions. In particular, one can completely describe the G_K-representation $H^n(X_{\overline{K},et}, \mathbf{Q}_p)$ in terms of the linear algebra data on the de Rham side, thus facilitating calculations. We will not be discussing any of these comparison theorems in this lecture series, and refer the reader to [BMS2, §1.1] for more information.

Remark 1.2.6 (Open and singular varieties)**.** Theorem 1.1.7 has a natural extension to arbitrary varieties X/K. In this case, the correct statement of the decomposition is:

$$H^n(X_{\overline{K},et}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \simeq \bigoplus_{i+j=n} \mathrm{gr}^j_F H^{i+j}_{dR}(X/K) \otimes_K \mathbf{C}_p(-j),$$

where gr^j_F denotes the j-th graded piece for the Hodge filtration on $H^n_{dR}(X/K)$ constructed by Deligne's theory of mixed Hodge structures [De1, De2]. This result falls most naturally out of the recent approach of Beilinson [Be] to the p-adic comparison theorems based on vanishing theorems for the h-topology; we shall not discuss such extensions further in these notes.

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2. Lecture 2: The Hodge-Tate decomposition for abelian schemes

The main goal for this section is to introduce, in the case of abelian varieties with good reduction, certain "large" constructions that are generally useful in Hodge-Tate theory. Our hope is that encountering these "large" objects in a relatively simple setting will help demystify them.

2.1 The statement

Fix a finite extension $K/\mathbf{Q}_p$, a completed algebraic closure $K \hookrightarrow C$, and an abelian scheme $\mathcal{A}/\mathcal{O}_K$ with generic fiber A. Our goal is to sketch a proof of the following result:

Theorem 2.1.1. *There exists a canonical isomorphism*

$$H^1(A_C, C) := H^1(A_{C,et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} C \simeq \big(H^1(A, \mathcal{O}_A) \otimes_K C\big) \oplus \big(H^0(A, \Omega^1_{A/K}) \otimes_K C(-1)\big).$$

In fact, we shall not construct the complete decomposition. Instead, we shall construct a map

$$\alpha_A : H^1(A, \mathcal{O}_A) \otimes_K C \to H^1(A_C, C)$$

using inspiration from the perfectoid theory (as well as [Be]), and a map

$$\beta_A : H^0(A, \Omega^1_{A/K}) \otimes_K C(-1) \to H^1(A_C, C)$$

exploiting the arithmetic of the base field K, following an idea of Fontaine [Fo1]. We can then put these together to get the map

$$\gamma_A = \alpha_A \oplus \beta_A : \big(H^1(A, \mathcal{O}_A) \otimes_K C\big) \oplus \big(H^0(A, \Omega^1_{A/K}) \otimes_K C(-1)\big) \to H^1(A_C, C),$$

that induces the Hodge-Tate decomposition.

Remark 2.1.2 (Reminders on abelian varieties)**.** The following facts about the cohomology of abelian varieties will be used below.

1. Write $T_p(A) := \lim A[p^n](C)$ for the p-adic Tate module of A. Then there is a natural identification of $H^*(A, C) \simeq H^*(T_p(A), C)$, where $H^*(T_p(A), C)$ denotes the continuous group cohomology of the profinite group $T_p(A)$ with coefficients in the topological ring C; this essentially comes down to the assertion that an abelian variety of dimension g over $\mathbf{C}$ is homeomorphic to $(S^1)^{2g}$. In particular, since $T_p(A) \cong \mathbf{Z}_p^{2g}$, one calculates that $H^*(A, C)$ is an exterior algebra on $H^1(A, C) \simeq T_p(A)^\vee \otimes C$.

2. The $\mathcal{O}_K$-module $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is free of rank $g = \dim(A)$. In fact, this module is canonically identified with the Lie algebra of the dual abelian scheme $\mathcal{A}^\vee$. Moreover, the cohomology ring $H^*(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is an exterior algebra on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ via cup products. In particular, all cohomology groups of $\mathcal{O}_A$ are torsionfree.

2.2 The perfectoid construction of the map α_A

The discussion in this section is geometric, so we work with the abelian $\mathcal{O}_C$-scheme $\mathcal{A}_{\mathcal{O}_C}$ directly. Write $\mathcal{A}_n = \mathcal{A}_{\mathcal{O}_C}$ for each $n \geq 0$, and consider the tower

$$... \to \mathcal{A}_{n+1} \xrightarrow{[p]} \mathcal{A}_n \xrightarrow{[p]} ... \xrightarrow{[p]} \mathcal{A}_0 := \mathcal{A}_{\mathcal{O}_C}$$

of multiplication by p maps on the abelian scheme $\mathcal{A}_{\mathcal{O}_C}$. Write $\mathcal{A}_\infty := \lim \mathcal{A}_n$ for the inverse limit of this tower, and $\pi : \mathcal{A}_\infty \to \mathcal{A}_0$ for the resulting map to the bottom of the tower; this inverse limit exists as multiplication by p is a finite map on $\mathcal{A}_{\mathcal{O}_C}$ (see [SP, Tag 01YX]), and its cohomology with reasonable coefficients (such as the structure sheaf) can be calculated as the direct limit of the cohomologies of the $\mathcal{A}_n$'s.

Now obseve that translating by p^n-torsion points gives an action of $\mathcal{A}[p^n](\mathcal{O}_C) \simeq A[p^n](C)$ on the map $\mathcal{A}_n \to \mathcal{A}_0$; here we use the valuative criterion of properness for the identification $\mathcal{A}[p^n](\mathcal{O}_C) \simeq A[p^n](C)$. Taking inverse limits in n, we obtain an action of $T_p(A)$ on the map π. Taking pullbacks, we obtain a map

$$H^*(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to H^*(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}).$$

Due to the presence of the group action, the image of this map is contained in the $T_p(A)$-invariants of the target. Thus, we can view preceding map as a map

$$H^*(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to H^0(T_p(A), H^*(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})),$$

where we use the notation $H^0(G, -)$ for the functor of taking G-invariants for a group G. Deriving this story, we obtain a map

$$\Psi : R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to R\Gamma_{conts}(T_p(A), R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})), \tag{2.1}$$

where $R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ denotes the cohomology of the structure sheaf on $\mathcal{A}$ and $R\Gamma_{conts}(T_p(A), -)$ denotes continuous group cohomology theory for the profinite group $T_p(A)$, both in the sense of derived categories (see [We, §10] for a quick introduction). To proceed further, we observe the following vanishing theorem:

Proposition 2.2.1. *The canonical map $\mathcal{O}_C \to R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$ induces an isomorphism modulo any power of p, and hence after p-adic completion.*

Proof. As $\mathcal{A}$ is an abelian scheme, its cohomology ring $H^*(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is an exterior algebra on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. Moreover, multiplication by an integer N on $\mathcal{A}$ induces multiplication by N on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. By combining these observations with the formula

$$H^i(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}) \simeq \operatorname*{colim}_n H^i(\mathcal{A}_n, \mathcal{O}_{\mathcal{A}_n}) \simeq \operatorname*{colim}_{[p]^*} H^i(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}}),$$

we learn that $H^i(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$ is the constants $\mathcal{O}_C$ if $i = 0$, and $H^i(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}})[\frac{1}{p}]$ for $i > 0$. In particular, working modulo any power of p, the latter vanishes, so we get the claim. ☐

Thus, after p-adic completion, the map Ψ gives a map

$$\widehat{\Psi} : R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to R\Gamma_{conts}(T_p(A), \mathcal{O}_C).$$

On the other hand, as abelian varieties are $K(\pi, 1)$'s, we can interpret the preceding map as a map

$$\widehat{\Psi} : R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to R\Gamma(A_C, \mathcal{O}_C).$$

In particular, applying H^1 and inverting p, we get a map

$$H^1(A, \mathcal{O}_A) \to H^1(A_C, C),$$

which then linearizes to the promised map

$$\alpha_A : H^1(A, \mathcal{O}_A) \otimes C \to H^1(A_C, C).$$

Remark 2.2.2 (Perfectoid abelian varieties)**.** Given any affine open $U \subset \mathcal{A}_{\mathcal{O}_C}$, write $U_\infty \subset \mathcal{A}_\infty$ for its inverse image. Then the p-adic completion R of $\mathcal{O}(U_\infty)$ is an *integral perfectoid* $\mathcal{O}_C$*-algebra*, i.e., R is p-adically complete and p-torsionfree, and the Frobenius induces an isomorphism $R/p^{\frac{1}{p}} \simeq R/p$. In particular, the generic fiber A_∞ of $\mathcal{A}_\infty$ gives a perfectoid space. In fact, the map $A_\infty \to A_C$ is a pro-étale $T_p(A)$-torsor. This construction may be viewed as the analog for abelian varieties of the perfectoid torus from Example 3.2.4 below.

2.3 Fontaine's construction of the map β_A

For Fontaine's construction, we need the following fact about the arithmetic of p-adic fields:

Theorem 2.3.1 (Differential forms on $\mathcal{O}_C$)**.** *Write Ω for the Tate module of $\Omega^1_{\mathcal{O}_C/\mathcal{O}_K}$. This $\mathcal{O}_C$-module is free of rank 1. Moreover, there is a Galois equivariant isomorphism $C(1) \simeq \Omega[\frac{1}{p}]$.*

Construction of the map giving the isomorphism. Consider the $d\log$ map

$$\mu_{p^\infty}(\mathcal{O}_C) \subset \mathcal{O}_C^* \to \Omega^1_{\mathcal{O}_C/\mathcal{O}_K}$$

given by $f \mapsto \frac{df}{f}$. On passage to Tate modules and linearizations, this gives a map

$$T_p(\mu_{p^\infty}(\mathcal{O}_C)) \otimes_{\mathbf{Z}_p} \mathcal{O}_C = \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathcal{O}_C = \mathcal{O}_C(1) \to \Omega.$$

Fontaine proves this map is injective with torsion cokernel, giving $C(1) \simeq \Omega[\frac{1}{p}]$; see [Fo1, §1] for Fontaine's proof, and [Be, §1.3] for a slicker (but terse) argument using the cotangent complex. □

Remark 2.3.2 (The cotangent complex of $\mathcal{O}_C$)**.** Theorem 2.3.1 also extends to the cotangent complex after a shift: one has $\widehat{L_{\mathcal{O}_C/\mathbf{Z}_p}} \simeq \Omega[1]$, where the completion on the left side is the derived p-adic completion. Although this can be deduced directly from Theorem 2.3.1, we do not explain this here; instead, we refer to Remark 3.1.12 where a more general statement is proven. This assertion will be useful later in constructing the Hodge-Tate filtration.

In particular, this result helps connect the Tate twist $C(1)$ (which lives on the Galois side of the story) to differential forms (which lie on the de Rham side). Using this, Fontaine's idea for constructing the map β_A is to pullback differential forms on $\mathcal{A}$ to those on $\mathcal{O}_C$ using points in $\mathcal{A}(\mathcal{O}_C)$. More precisely, this pullback gives a pairing

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K}) \otimes \mathcal{A}(\mathcal{O}_C) \to \Omega^1_{\mathcal{O}_C/\mathcal{O}_K}.$$

Passing to p-adic Tate modules, this gives a pairing

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K}) \otimes T_p(A) \to \Omega.$$

Using the identification $T_p(A) \simeq H^1(A_{C,et}, \mathbf{Z}_p)^\vee$, this gives a map

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K}) \to H^1(A_{C,et}, \mathbf{Z}_p) \otimes \Omega.$$

Inverting p and using Theorem 2.3.1, we get the map

$$H^0(A, \Omega^1_{A/K}) \to H^1(A_C, C)(1).$$

Linearizing and twisting gives the desired map

$$\beta_A : H^0(A, \Omega^1_{A/K}) \otimes C(-1) \to H^1(A_C, C).$$

2.4 Conclusion

Taking direct sums of the previous two constructions gives the map $\gamma_A = \alpha_A \oplus \beta_A$

$$\gamma_A : \left(H^1(A, \mathcal{O}_A) \otimes C\right) \oplus \left(H^0(A, \Omega^1_{A/K})(-1) \otimes C\right) \to H^1(A_C, C).$$

Each of the parenthesized summands on the left has dimension g, while the target has dimension on $2g$. Thus, to show γ_A is an isomorphism, it is enough to show injectivity. Moreover, by Tate's calculations in Theorem 1.1.8, it is enough to show that α_A and β_A are separately injective: the map γ_A is Galois equivariant, and the two summands have different Galois actions, so they cannot talk to each other. For Fontaine's map β_A, this follows by a formal group argument as it is enough to check the corresponding assertion for the formal group of $\mathcal{A}$. For the map α_A, we are not aware of a direct argument that does not go through one of the proofs of the p-adic comparison theorems. For lack of space, we do not give either argument here.

3. Lecture 3: The Hodge-Tate decomposition in general

Let C be a complete and algebraically closed extension of $\mathbf{Q}_p$. Let X/C be a smooth rigid-analytic space[1]. Our goal is to relate the étale cohomology of X to differential forms. More precisely, setting

$$H^n(X, C) := H^n(X_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} C,$$

we want to prove the following result:

Theorem 3.0.1 (Hodge-Tate spectral sequence)**.** *Assume X is proper. Then there exists an E_2-spectral sequence*

$$E_2^{i,j} : H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X, C).$$

If X is defined over a discretely valued subfield K of C, then all differentials are canonically 0, and we obtain Theorem 1.1.7.

The starting point of this relation between étale cohomology and differential forms is the completed structure sheaf $\widehat{\mathcal{O}_X}$ on the pro-étale site X_{proet} of X; these objects are defined in §3.2. To a first approximation, objects of X_{proet} may be viewed as towers $\{U_i\}$ of finite étale covers with $U_0 \to X$ étale, and $\widehat{\mathcal{O}_X}$ is the sheaf which assigns to such a tower the completion of the direct limit of the rings of analytic functions on the U_i's. In particular, this is a sheaf of C-algebras. The following comparison theorem [Sc2, Theorem 5.1] relates the cohomology of $\widehat{\mathcal{O}_X}$ to more topological invariants:

Theorem 3.0.2 (Primitive comparison theorem)**.** *If X is proper, then the inclusion $C \subset \widehat{\mathcal{O}_X}$ of the constants gives an isomorphism*

$$H^*(X, C) \simeq H^*(X_{proet}, \widehat{\mathcal{O}_X}).$$

Thus, to prove Theorem 3.0.1, it suffices to work with $H^n(X_{proet}, \widehat{\mathcal{O}_X})$ instead of $H^n(X, C)$, thus putting both sides of Theorem 3.0.1 into the realm of coherent cohomology. To proceed further, we recall that there is a canonical projection map

$$\nu : X_{proet} \to X_{et}$$

[1] All results in this section are due to Scholze unless otherwise specified. When X is a smooth proper variety, some of the results were proven by Faltings [Fa1, Fa4] in a different language. When discussing the étale cohomology of adic spaces, we are implicitly using Huber's theory [Hu2]. When X arises as the analytification of an algebraic variety Y, Huber's étale cohomology groups agree with those of Y, so we can draw consequences for the algebraic theory as well.

from the pro-étale site of X to the étale site of X; recall that a morphism of sites goes in the other direction from the underlying functor of categories, and the latter for ν simply captures the fact that an étale morphism is pro-étale. Theorem 3.0.1 arises from the Leray spectral sequence for ν using the following:

Theorem 3.0.3 (Hodge-Tate filtration: local version)**.** *There is a canonical isomorphism* $\Omega^1_{X/C}(-1) \simeq R^1\nu_*\widehat{\mathcal{O}_X}$*. Taking products, this gives isomorphisms* $\Omega^j_{X/C}(-j) \simeq R^j\nu_*\widehat{\mathcal{O}_X}$*.*

In the rest of this lecture, we will sketch a proof of this result. More precisely, §3.2 contains some reminders on the pro-étale site, especially its locally perfectoid nature. This is then used in §3.4 to construct the map giving the isomorphism of Theorem 3.0.3; this construction relies on the cotangent complex (whose basic theory is reviewed in §3.1), and differs from that in [Sc2]. Once the map has been constructed, we check that it is an isomorphism in §3.5 using the almost acyclicity of the structure sheaf for affinoid perfectoid spaces.

Remark 3.0.4 (Hodge and Hodge-Tate filtrations)**.** The differentials in Theorem 3.0.1 are, in fact, always 0, and thus one always has *some* Hodge-Tate decomposition as in Theorem 1.1.7. This result is explained in [BMS2, Theorem 13.12] and relies on the work of Conrad-Gabber [CG] on spreading out rigid-analytic families to reduce to the corresponding assertion over discretely valued fields. However, these differentials are not *canonically* 0. More precisely, the complex $R\nu_*\widehat{\mathcal{O}_X}$ is not a direct sum of its cohomology sheaves. Concretely, when X is an abelian variety, one has a canonical map $H^1(X, \mathcal{O}_X) \to H^1(X, C)$ as explained in §2, giving the *Hodge-Tate filtration* on $H^1(X, C)$; however, one cannot choose a splitting $H^1(X, C) \to H^1(X, \mathcal{O}_X)$ in a manner that is compatible in families of abelian varieties. Instead, the variation of the Hodge-Tate filtration in a family of abelian varieties provides a highly non-trivial and interesting invariant of the family: the Hodge-Tate period map from [Sc4, §III.3].

The above discussion is analogous to the following (perhaps more familiar and) more classical story over **C** (see [Vo, §10]): even though the Hodge-to-de Rham spectral sequence always degenerates for a smooth projective variety, one cannot choose a Hodge decomposition for smooth projective varieties that varies holomorphically in a family. Instead, it is the Hodge *filtration* on de Rham cohomology that varies holomorphically. In fact, the variation of this filtration in a family of smooth projective varieties provides an extremely important invariant of the family: the period map to the classifying space for Hodge structures.

Remark 3.0.5 (The first obstruction to splitting the Hodge-Tate filtration)**.** Consider the complex $K := \tau^{\leq 1} R\nu_*\widehat{\mathcal{O}_X}$. This complex has 2 nonzero cohomology sheaves (identified by Theorem 3.0.3), and thus sits in an exact triangle

$$\mathcal{O}_X \to K \to \Omega^1_{X/C}(-1)[-1].$$

The boundary map for this exact triangle is a map

$$\Omega^1_{X/C}(-1)[-1] \to \mathcal{O}_X[1],$$

and thus gives an element of $\mathrm{ob}_X \in \mathrm{Ext}^2_X(\Omega^1_{X/C}, \mathcal{O}_X(1))$. To understand this element better, recall the exact sequence

$$0 \to \ker(\theta)/\ker(\theta)^2 \simeq \mathcal{O}_C(1) \to A_{\mathrm{inf}}(\mathcal{O}_C)/\ker(\theta)^2 \xrightarrow{\overline{\theta}} A_{\mathrm{inf}}(\mathcal{O}_C)/\ker(\theta) \simeq \mathcal{O}_C \to 0.$$

The map $\overline{\theta}$ is a non-trivial Galois equivariant square-zero extension of the commutative ring $\mathcal{O}_C$ by $\mathcal{O}_C(1)$, and the same holds true after inverting p. One can then show (using the method of §3.4) that ob_X is precisely the obstruction to lifting X across the thickening $\overline{\theta}[\frac{1}{p}]$, thus giving some geometric meaning to the non-canonicity of the Hodge-Tate decomposition mentioned in Remark 3.0.4[2]. The characteristic p analog of this is the Deligne-Illusie result, recalled next.

Remark 3.0.6 (Deligne-Illusie obstructions to liftability)**.** Remark 3.0.5 is analogous to a more classical picture from [DI], which we recall. Let k be a perfect field of characteristic p, and let X/k be a smooth k-scheme. Consider the truncated de Rham complex $K := \tau^{\leq 1}\Omega^\bullet_{X/k}$. Note that the differentials in the de Rham complex $\Omega^\bullet_{X/k}$ are linear over k and the p-th powers $\mathcal{O}_X^p$ of functions on X. Thus, we may view $\Omega^\bullet_{X/k}$ (and thus K) as a complex of coherent sheaves on the Frobenius twist $X^{(1)}$ of X relative to k. By a theorem of Cartier, one has $\mathcal{H}^i(\Omega^\bullet_{X/k}) \simeq \Omega^i_{X^{(1)}/k}$. Thus, the complex K sits in an exact triangle

$$\mathcal{O}_{X^{(1)}} \to K \to \Omega^1_{X^{(1)}/k}[-1].$$

The boundary map for this triangle is a map

$$\Omega^1_{X^{(1)}/k}[-1] \to \mathcal{O}_{X^{(1)}}[1],$$

and can thus be viewed as an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{X^{(1)}}(\Omega^1_{X^{(1)}/k}, \mathcal{O}_{X^{(1)}})$. One of the main observations of [DI] is that ob_X is precisely the obstruction to lifting the k-scheme $X^{(1)}$ along the square-zero extension $W_2(k) \to k$ of k.

3.1 The cotangent complex and perfectoid rings

We recall the construction and basic properties of the cotangent complex; much more thorough accounts can be found in [Qu2, Ill1, Ill2] and [SP, Tag 08P5]. Once the basics have been introduced, we shall explain some applications to the perfectoid theory; the key point is that maps between perfectoids are formally étale in a strong sense, and this perspective helps conceptualize certain results about them (such as the tilting correspondence and Fontaine's calculation of differential forms in Theorem 2.3.1) better. We begin with the following construction from non-abelian homological algebra:

Construction 3.1.1 (Quillen)**.** For any ring A and a set S, we write $A[S]$ for the polynomial algebra over A on a set of variables x_s indexed by $s \in S$. The functor $S \mapsto A[S]$ is left adjoint to the forgetful functor from A-algebras to sets. In particular, for any A-algebra B, we have a canonical map $\eta_B : A[B] \to B$, which is evidently surjective. Repeating the construction, we obtain two natural A-algebra maps $\eta_{A[B]}, A[\eta_B] : A[A[B]] \to A[B]$. Iterating this process allows one to define a simplicial A-algebra $P^\bullet_{B/A}$ augmented over B that looks like

$$P^\bullet_{B/A} := \Big(...A[A[A[B]]] \rightrightarrows A[A[B]] \rightrightarrows A[B]\Big) \longrightarrow B.$$

This map is a resolution of B in the category of simplicial A-algebras, and is called the canonical simplicial A-algebra resolution of B; concretely, this implies that the chain complex underlying $P^\bullet_{B/A}$ (obtained by

[2]Forthcoming work of Conrad-Gabber [CG] shows that this obstruction class is always 0, at least when X is assumed to be proper. Nevertheless, this class admits an integral analog, which can be nonzero; see Remark 3.1.15.

taking an alternating sum of the face maps as differentials) is a free resolution of B over A. Slightly more precisely, there is a model category of simplicial A-algebras, and the factorization $A \to P^\bullet_{B/A} \to B$ provides a functorial cofibrant replacement of B, and can thus be used to calculate non-abelian derived functors. We do not discuss this theory here, and will take certain results (such as the fact that such polynomial A-algebra resolutions are unique up to a suitable notion of homotopy) as blackboxes; a thorough discussion, in the language of model categories, can be found in [Qu1].

Using the previous construction, the main definition is:

Definition 3.1.2 (Quillen)**.** For any map $A \to B$ of commutative rings, we define its cotangent complex $L_{B/A}$, which is a complex of B-modules and viewed as an object of the derived category $D(B)$ of all B-modules, as follows: set $L_{B/A} := \Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$, where $P^\bullet \to B$ is a simplicial resolution of B by polynomial A-algebras. Here we view the simplicial B-module $\Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ as a B-complex by taking an alternating sum of the face maps as a differential.

For concreteness and to obtain a strictly functorial theory, one may choose the canonical resolution $P^\bullet_{B/A}$ in the definition above. However, in practice, just like in homological algebra, it is important to allow the flexibility of changing resolutions without changing $L_{B/A}$ (up to quasi-isomorphism). The following properties can be checked in a routine fashion, and we indicate a brief sketch of the argument:

1. *Polynomial algebras.* If B is a polynomial A-algebra, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$: this follows because any two polynomial A-algebra resolutions of B are homotopic to each other, so we may use the constant simplicial A-algebra with value B to compute $L_{B/A}$.

2. *Künneth formula.* If B and C are flat A-algebras, then $L_{B \otimes_A C/A} \simeq L_{B/A} \otimes_A C \oplus B \otimes_A L_{C/A}$: this reduces to the case of polynomial algebras by passage to resolutions. The flatness hypothesis gets used in concluding that if $P^\bullet \to B$ and $Q^\bullet \to C$ are polynomial A-algebra resolutions, then $P^\bullet \otimes_A Q^\bullet \to B \otimes_A C$ is also a polynomial A-algebra resolution. (In fact, this reasoning shows that the flatness hypotheses can be relaxed to the assumption $\mathrm{Tor}^A_{>0}(B, C) = 0$ provided one uses derived tensor products of chain complexes in the formula above.)

3. *Transitivity triangle.* Given a composite $A \to B \to C$ of maps, we have a canonical exact triangle

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$$

   in $D(C)$. To prove this, one first settles the case where $A \to B$ and $B \to C$ are polynomial maps (which reduces to a classical fact in commutative algebra). The general case then follows by passage to the canonical resolutions as the exact sequences constructed in the previous case were functorial.

4. *Base change.* Given a flat map $A \to C$ and an arbitrary map $A \to B$, we have $L_{B/A} \otimes_A C \simeq L_{B \otimes_A C/C}$. Again, one first settles the case of polynomial rings, and then reduces to this by resolutions, using flatness to reduce a derived base change to a classical one. (Again, this reasoning shows that the flatness hypothesis can be relaxed to the assumption $\mathrm{Tor}^A_{>0}(B, C) = 0$ provided one uses derived tensor products of chain complexes in the formula above.)

5. *Vanishing for étale maps.* We claim that if $A \to B$ is étale, then $L_{B/A} \simeq 0$. For this, assume first that $A \to B$ is a Zariski localization. Then $B \otimes_A B \simeq B$, so (2) implies that $L_{B/A} \oplus L_{B/A} \simeq L_{B/A}$ via the sum map. This immediately gives $L_{B/A} = 0$ for such maps. In general, as $A \to B$ is étale, the multiplication map $B \otimes_A B \to B$ is a Zariski localization, and thus $L_{B/B \otimes_A B} \simeq 0$. By the transitivity triangle for $B \xrightarrow{i_1} B \otimes_A B \to B$, this yields $L_{B \otimes_A B/B} \otimes_{B \otimes_A B} B \simeq 0$. But, by (4), we have $L_{B \otimes_A B/B} \simeq L_{B/A} \otimes_A B$, so the base change of $L_{B/A}$ along $A \to B \to B \otimes_A B \to B$ vanishes. The latter is just the structure map $A \to B$, so $L_{B/A} \otimes_A B \simeq 0$. The standard map $L_{B/A} \to L_{B/A} \otimes_A B$ has a section coming from the B-action on $L_{B/A}$, so $L_{B/A} \simeq 0$.

6. *Étale localization.* If $B \to C$ is an étale map of A-algebras, then $L_{B/A} \otimes_B C \simeq L_{C/A}$: this follows from (3) and (5) as $L_{C/B} \simeq 0$.

7. *Relation to Kähler differentials.* For any map $A \to B$, we have $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$. This can be shown directly from the definition.

8. *Smooth algebras.* If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$. By (6), there is a natural map $L_{B/A} \to \Omega^1_{B/A}[0]$. To show this is an isomorphism, we may work locally on A by (6). In this case, there is an étale map $B' := A[x_1, ..., x_n] \to B$. We know that $L_{B'/A} \simeq \Omega^1_{B'/A}[0]$ by (1) and $L_{B/B'} \simeq 0$ by (6). By (3), it follows that $L_{B/A} \simeq L_{B'/A} \otimes_{B'} B \simeq \Omega^1_{B/A}[0]$.

We give an example of the use of these properties in a computation.

Example 3.1.3 (Cotangent complex for a complete intersection)**.** Let R be a ring, let $I \subset R$ be an ideal generated by a regular sequence, and let $S = R/I$. Then we claim that $L_{S/R} \simeq I/I^2[1]$. In particular, this is a *perfect complex*, i.e., quasi-isomorphic to a finite complex of finite projective modules. To see this isomorphism, consider first the case $R = \mathbf{Z}[x_1, ..., x_r]$ and $I = (x_i)$. In this case, $S = \mathbf{Z}$, and the transitivity triangle for $\mathbf{Z} \to R \to S$ collapses to give $L_{S/R} \simeq \Omega^1_{R/\mathbf{Z}} \otimes_R S[1] \simeq I/I^2[1]$, where the isomorphism $I/I^2 \to \Omega^1_{R/\mathbf{Z}} \otimes_R S$ is defined by $f \mapsto df$. For general R, once we choose a regular sequence $f_1, ... f_r$ generating I, we have a pushout square of commutative rings

$$\begin{array}{ccc} \mathbf{Z}[x_1, ..., x_r] & \xrightarrow{x_i \mapsto f_i} & R \\ \downarrow{\scriptstyle x_i \mapsto 0} & & \downarrow \\ \mathbf{Z} & \longrightarrow & S. \end{array}$$

As the f_i's form a regular sequence, this is also a derived pushout square, i.e., $\mathrm{Tor}_{>0}^{\mathbf{Z}[x_1,...,x_r]}(R, \mathbf{Z}) = 0$. Base change for the cotangent complex implies that $L_{S/R} \simeq L_{\mathbf{Z}/\mathbf{Z}[x_1,...,x_r]} \otimes_{\mathbf{Z}} S \simeq I/I^2[1]$.

Assume now that with R, I, S as above, the ring R is smooth over a base ring k. Then $L_{R/k} \simeq \Omega^1_{R/k}$ is locally free. The transitivity triangle for $k \to R \to S$ then tells us that $L_{S/k}$ is computed by the following 2-term complex of locally free S-modules:

$$I/I^2 \xrightarrow{f \mapsto df} \Omega^1_{R/k} \otimes_R S.$$

Here the identification of the differential involves unraveling some of the identifications above. In particular, $L_{S/k}$ is also a perfect complex. Conversely, it is a deep theorem of Avramov (conjectured by Quillen) that if k is a field and $L_{S/k}$ is perfect for a finite type k-algebra S, then S is a complete intersection.

Remark 3.1.4 (Naive cotangent complex)**.** For most applications in algebraic geometry and number theory (including all that come up in these notes), it suffices to work with the truncation $\tau^{\geq -1} L_{B/A}$. This is a complex of B-modules with (at most) two non-zero cohomology groups in degrees -1 and 0. It can be constructed explicitly using a presentation: if $A \to B$ factors as $A \to P \to B$ with $A \to P$ a polynomial algebra and $P \to B$ surjective with kernel I, then we have

$$\tau^{\geq -1} L_{B/A} := \Big(I/I^2 \xrightarrow{f \mapsto df} \Omega^1_{P/A} \otimes_P B \Big).$$

This object is sometimes called the *naive cotangent complex*, and its basic theory is developed in [SP, 00S0]. Despite the elementary definition, it is sometimes awkward to work with the truncated object, so we stick to the non-truncated version in these notes.

The main reason to introduce the cotangent complex is that it controls deformation theory in complete generality, analogous to how the tangent bundle controls deformations of smooth varieties. In particular, the following consequence is relevant to us:

Theorem 3.1.5 (Deformation invariance of the category of formally étale algebras)**.** *For any ring A, write $\mathcal{C}_A$ for the category of flat A-algebras B such that $L_{B/A} \simeq 0$. Then for any surjective map $\tilde{A} \to A$ with nilpotent kernel, base change induces an equivalence $\mathcal{C}_{\tilde{A}} \simeq \mathcal{C}_A$. In other words, every $A \to B$ in $\mathcal{C}_A$ lifts uniquely (up to unique isomorphism) to $\tilde{A} \to \tilde{B}$ in $\mathcal{C}_{\tilde{A}}$.*

Any étale A-algebra B is an object of $\mathcal{C}_A$; conversely, every finitely presented A-algebra B in $\mathcal{C}_A$ is étale over A (see [SP, Tag 0D12] for a more general assertion). Thus, for such maps, Theorem 3.1.5 captures the topological invariance of the étale site (see [SP, Tag 04DZ]). However, the finite presentation hypothesis is too restrictive for applications in the perfectoid theory; instead, the following class of examples is crucial:

Proposition 3.1.6. *Assume A has characteristic p. Let $A \to B$ be a flat map that is relatively perfect, i.e., the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A,F_A} A \to B$ is an isomorphism. Then $L_{B/A} \simeq 0$.*

Proof. We first claim that for any A-algebra B, the relative Frobenius induces the 0 map $L_{F_{B/A}} : L_{B^{(1)}/A} \to L_{B/A}$: this is clear when B is a polynomial A-algebra (as $d(x^p) = 0$), and thus follows in general by passage to the canonical resolutions. Now if $A \to B$ is relatively perfect, then $L_{F_{B/A}}$ is also an isomorphism by functoriality. Thus, the 0 map $L_{B^{(1)}/A} \to L_{B/A}$ is an isomorphism, so $L_{B/A} \simeq 0$. □

This leads to the following conceptual description of the Witt vector functor:

Example 3.1.7 (Witt vectors via deformation theory)**.** Let R be a perfect ring of characteristic p. Then R is relatively perfect over $\mathbf{Z}/p$. Proposition 3.1.6 tells us that $L_{R/\mathbf{F}_p} \simeq 0$, so Theorem 3.1.5 implies that R has a flat lift R_n to $\mathbf{Z}/p^n$ for any $n \geq 1$, and that this lift is unique up to unique isomorphism. In fact, this lift is simply given by the Witt vector construction $W_n(R)$. Setting $W(R) = \lim_n W_n(R)$ gives the Witt vectors of R, which can also be seen as the unique p-adically complete p-torsionfree $\mathbf{Z}_p$-algebra lifting R. This perspective also allows one to see some additional structures on $W(R)$. For example, the map $R \to R$ of multiplicative monoids lifts uniquely across any map $W_n(R) \to R$: the monoid R is uniquely p-divisible, while the fiber over $1 \in R$ of $W_n(R) \to R$ is p-power torsion. Explicitly, one simply sends $r \in R$ to $\widetilde{r_n}^{p^n}$, where $\widetilde{r_n} \in W_n(R)$ denotes some lift of $r_n := r^{\frac{1}{p^n}}$. The resulting multiplicative maps $R \to W_n(R)$ and $R \to W(R)$ are called the Teichmüller lifts, and denoted by $r \mapsto [r]$.

Remark 3.1.8 (Fontaine's A_{inf} and the map θ)**.** Fix a ring A and a map $A \to B$ in $\mathcal{C}_A$. With a bit more care in analyzing deformation theory via the cotangent complex (see [SP, Tag 0D11]), one can show the following lifting feature: if $C' \to C$ is a surjective A-algebra map with a nilpotent kernel, then every A-algebra map $B \to C$ lifts unique to an A-algebra map $B \to C'$. In particular, given a p-adically complete $\mathbf{Z}_p$-algebra C, a perfect ring D, and a map $D \to C/p$, we obtain a unique lift $W_n(D) \to C/p^n$ of the composition $W_n(D) \to D \to C/p$ for each n. Taking limits, we obtain unique map $W(D) \to C$ lifting the map $W(D) \to C/p$ arising via $W(D) \to D \to C/p$. Applying this in a universal example of such a D for a given C, we obtain Fontaine's map θ from [Fo2] via abstract nonsense:

Proposition 3.1.9. *Given any p-adically complete ring R, the canonical projection map $\overline{\theta} : R^\flat := \lim_\phi R/p \to R/p$ lifts to a unique map $\theta : A_{inf}(R) := W(R^\flat) \to R$.*

Note that the map $\overline{\theta} : R^\flat \to R/p$ is surjective exactly when R/p is *semiperfect*, i.e., has a surjective Frobenius. In this case, the map θ is also surjective by p-adic completeness.

We next explain the relevance of these ideas to the perfectoid theory. As the definition of perfectoid algebras varies somewhat depending on context, we define the notion we need (see [BMS2, §3.2] for more on such rings), using the map θ introduced above:

Definition 3.1.10. A ring R is *integral perfectoid* if R is π-adically complete for some element π with $\pi^p \mid p$, the ring R/p has a surjective Frobenius, and the kernel of $A_{inf}(R) := W(R^\flat) \to R$ is principal.

Note that being integral perfectoid is a property of the ring R as an abstract ring (as opposed to a topological ring, or an algebra over some other fixed ring). Important examples include the rings of integers of perfectoid fields (in the sense of [Sc1, Definition 3.1]), and any perfect ring of characteristic p. In fact, if C is a perfectoid field of characteristic 0, then a p-adically complete and p-torsionfree $\mathcal{O}_C$-algebra R is integral perfectoid exactly when the map $\mathcal{O}_C/p \to R/p$ is relatively perfect in the sense of Proposition 3.1.6.

Remark 3.1.11 (Tilting)**.** For an integral perfectoid ring R, the map $\theta : A_{inf}(R) \to R$ from Proposition 3.1.9 fits into the following commutative diagram

$$\begin{array}{ccc} A_{inf}(R) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ R^\flat & \xrightarrow{\overline{\theta}} & R/p, \end{array}$$

where each map can be regarded as a pro-infinitesimal thickening of the target by the perfectoidness assumption. In particular, all 4 rings are pro-infinitesimal thickenings of R/p. Theorem 3.1.5 and Proposition 3.1.6 may then be used to prove half of the tilting correspondence from [Sc1, Theorem 5.2].

We make some remarks on the differential aspects of perfectoid rings.

Remark 3.1.12 (Formally étale nature of A_{inf} and differential forms)**.** Let A be a perfect ring of characteristic p. By Example 3.1.7, the map $\mathbf{Z}_p \to W(A)$ satisfies the following crucial feature: the cotangent complex $L_{A/\mathbf{F}_p} \simeq 0$, so the p-adic completion $\widehat{L_{W(A)/\mathbf{Z}_p}}$ vanishes by base change for cotangent complexes and Nakayama's lemma for p-adically complete complexes. By the transitivity triangle, for any $W(A)$-algebra R, we have $\widehat{L_{R/\mathbf{Z}_p}} \simeq \widehat{L_{R/W(A)}}$. Now specialize to the case where R is an integral perfectoid ring and $A = R^\flat$, with R viewed as an algebra over $W(A) = A_{inf}(R^\flat)$ via θ. Then we learn that

$$\widehat{L_{R/\mathbf{Z}_p}} \simeq \widehat{L_{R/A_{inf}(R^\flat)}}.$$

But the map $\theta : A_{inf}(R^\flat) \to R$ is a quotient by a nonzerodivisor in $A_{inf}(R^\flat)$ (see [BMS2, Lemma 3.10 (i)] for a proof). Using Example 3.1.3, this tells us that

$$\widehat{L_{R/\mathbf{Z}_p}} \simeq \ker(\theta)/\ker(\theta)^2[1].$$

In particular, this is a free R-module of rank 1. In the special case where $R = \mathcal{O}_C$ for a complete and algebraically closed extension $C/\mathbf{Q}_p$, this essentially recovers Fontaine's theorem 2.3.1; to arrive at the precise Galois module structure given in Theorem 2.3.1, one simply observes that if $\underline{\epsilon} := (1, \epsilon_p, \epsilon_{p^2}, ...) \in \mathcal{O}_C^\flat$ is a non-trivial compatible sequence of p-power roots of 1, then $\mu := [\underline{\epsilon}] - 1 \in \ker(\theta)$, and its image in $\ker(\theta)/\ker(\theta)^2$ spans a copy of $\mathcal{O}_C(1)$; the quotient $\Big(\ker(\theta)/\ker(\theta)^2\Big)/\mathcal{O}_C(1)$ is then torsion, and can be shown to be killed by $p^{\frac{1}{p-1}}$.

Remark 3.1.13 (Breuil-Kisin twists)**.** For future reference, we remark that the $\mathcal{O}_C$-module

$$\Omega := T_p(\Omega^1_{\mathcal{O}_C/\mathbf{Z}_p}) \simeq \widehat{L_{\mathcal{O}_C/\mathbf{Z}_p}}[-1] \simeq \ker(\theta)/\ker(\theta)^2$$

is a canonically defined invertible $\mathcal{O}_C$-module (as it is abstractly free of rank 1), and we shall write

$$M \mapsto M\{i\} := M \otimes_{\mathcal{O}_C} \Omega^{\otimes i}$$

for the corresponding twisting operation on $\mathcal{O}_C$-modules; when M carries a Galois action, so does the twist. These objects are called the *Breuil-Kisin twists* of M, and are related to the Tate twist via an inclusion $M(i) \subset M\{i\}$ for $i \geq 0$ with a torsion cokernel (see the construction following Theorem 2.3.1 for an explanation of the origin of this inclusion). Slightly more generally, the same discussion applies when $\mathcal{O}_C$ is replaced by an integral perfectoid ring R to define a twisting operation $M \mapsto M\{1\} := M \otimes_R \ker(\theta)/\ker(\theta)^2$ on R-modules (but one loses the analog of the inclusion $M(1) \subset M\{1\}$ available for $R = \mathcal{O}_C$).

One fruitful viewpoint on integral perfectoid rings is to view them as integral analogs of perfect rings: they share some of the miraculous properties of perfect characteristic p rings without themselves having characteristic p. This perspective leads one to predict certain results in mixed characteristic, and we explain how this plays out for the Deligne-Illusie theorem in the next two remarks.

Remark 3.1.14 (Deligne-Illusie, revisited)**.** Let k be a perfect field of characteristic p. Then $k \simeq W(k)/p$, so $L_{k/W(k)} \simeq k[1]$ by Example 3.1.3. Now consider a smooth k-algebra R. The transitivity triangle for $W(k) \to k \to R$ is

$$L_{k/W(k)} \otimes_k R \to L_{R/W(k)} \to L_{R/k}.$$

Using the smoothness of R and the previous computation of $L_{k/W(k)}$, this simplifies to

$$R[1] \to L_{R/W(k)} \to \Omega^1_{R/k}.$$

As this construction is functorial in R, we may sheafify it to obtain the following: for smooth k-scheme X, we have a functorial exact triangle

$$\mathcal{O}_X[1] \to L_{\mathcal{O}_X/W(k)} \to \Omega^1_{X/k}.$$

In particular, the boundary map for this triangle is

$$\Omega^1_{X/k} \to \mathcal{O}_X[2],$$

and can thus be identified as a class

$$\mathrm{ob}_{X/W(k)} \in \mathrm{Ext}^2(\Omega^1_{X/k}, \mathcal{O}_X).$$

Using the deformation-theoretic interpretation of the cotangent complex (and unravelling Example 3.1.3), one can show that $\mathrm{ob}_{X/W(k)}$ is precisely the obstruction to lifting X to $W_2(k)$. The main theorem of Deligne-Illusie [DI] is that the obstruction class ob_X constructed in Remark 3.0.6 via the de Rham complex coincides with $\mathrm{ob}_{X^{(1)}/W(k)}$; equivalently, the complex $L_{X^{(1)}/W}[-1]$ identifies with $\tau^{\leq 1}\Omega^\bullet_{X/k}$.

Remark 3.1.15 (The integral analog of Deligne-Illusie)**.** The analogy between perfect rings and integral perfectoid rings is strong enough that we can directly repeat the discussion of Remark 3.1.14 when the ring k is only assumed to be an integral perfectoid ring. In this case, we must replace $W(k)$ with $A_{inf}(k) = W(k^\flat)$ and the map $W(k) \to k$ with Fontaine's map $\theta : A_{inf}(k) \to k$. Given a smooth k-scheme X, the discussion in Remark 3.1.14 goes through (using Remark 3.1.12) to construct a class $\mathrm{ob}_{X/A_{inf}(k)} \in \mathrm{Ext}^1_X(\Omega^1_{X/k}, \mathcal{O}_X\{1\})$ from the complex $L_{X/A_{inf}(k)}$: it measures the failure to lift X across $\overline{\theta} : A_{inf}(k)/\ker(\theta)^2 \to k$. In this setting, the analog of the Deligne-Illusie theorem is then the subject of [BMS2, §8]; the rational version for $k = \mathcal{O}_C$ with C an algebraically closed perfectoid field of characteristic 0 is the identification of $L_{X/A_{inf}(k)}[-1][\frac{1}{p}]$ with $\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}_{X_C}}$, as alluded to in Remark 3.0.5.

The notation A_{inf} (and its cousin A_{crys}) were adopted for geometric reasons, as we briefly recall.

Remark 3.1.16 (Nomenclature of A_{inf} and A_{crys})**.** Let R be an integral perfectoid ring. By definition, the map $\overline{\theta} : R^\flat \to R/p$ is a projective limit of the maps $R/p \xrightarrow{\phi^n} R/p$; by the perfectoidness of R, each of these latter maps is a infinitesimal thickening (i.e., is surjective with nilpotent kernel). Thus, we may regard $R^\flat$ as a projective limit of infinitesimal thickenings on R. Moreover, by perfectness, $R^\flat \to R/p$ is the universal such object in characteristic p rings: for any other infinitesimal thickening $S \to R/p$ with S an $\mathbf{F}_p$-algebra, there is a unique map $R^\flat \to S$ factoring $\overline{\theta}$. As in Proposition 3.1.9, one then checks that $\theta : A_{inf}(R) \to R/p$ is also a projective limit of infinitesimal thickenings of R/p, and is the universal such object amongst all thickenings. Stated differently, $A_{inf}(R)$ is the global sections of the structure sheaf of the infinitesimal site for $\mathrm{Spec}(R/p)$ (see [Gro] for the infinitesimal and crystalline site); this is the origin of Fontaine's notation $A_{inf}(R)$ (which arose in the example $R = \mathcal{O}_{\mathbf{C}_p}$ first). Likewise, in this case, adjoining divided powers along the kernel of θ and p-adically completing produces Fontaine's period ring $A_{crys}(R)$, which comes equipped with a factorization $A_{inf}(R) \to A_{crys}(R) \to R$; one can then show that the map $A_{crys}(R) \to R$ realizes $A_{crys}(R)$ as the global sections of the structure sheaf on the crystalline site of $\mathrm{Spec}(R/p)$, once again explaining the notation.

3.2 Recollections on the pro-étale site

We now return to the setup at the start of the section: C is a complete and algebraically closed extension of $\mathbf{Q}_p$, and X/C is a smooth rigid-analytic space. Viewing X as an adic space, Scholze has attached its pro-étale site X_{proet} in [Sc2, §3] (see also lectures by Kedlaya and Weinstein[3]). A typical object here is a pro-object $U := \{U_i\}$ of X_{et} such that all transition maps $U_i \to U_j$ are finite étale covers for i, j large. Heuristically, one wishes to allow towers of finite étale covers of an open in X. The following class of objects in X_{proet} plays a crucial role:

Definition 3.2.1. An object $U := \{U_i\} \in X_{proet}$ is called *affinoid perfectoid* if it satisfies the following:

1. Each $U_i = \mathrm{Spa}(R_i, R_i^+)$ is affinoid.
2. Setting $R^+ := \widehat{\mathrm{colim}_i\, R_i^+}$ (where the completion is p-adic) and $R = R^+[\frac{1}{p}]$, the pair (R, R^+) is a perfectoid affinoid algebra.

For such an object U, we write $\widehat{U} := \mathrm{Spa}(R, R^+)$ for the corresponding perfectoid space.

For our purposes, the main reason to enlarge the étale site X_{et} to the pro-étale site X_{proet} is that the following theorem, stating roughly that there are enough affinoid perfectoid objects to cover any object, becomes true (see [Sc2, Corollary 4.7]):

Theorem 3.2.2 (Locally perfectoid nature of X_{proet})**.** *The collection of $U \in X_{proet}$ which are affinoid perfectoid form a basis for the topology.*

Remark 3.2.3. The construction of pro-étale site makes sense any noetherian adic space X over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Moreover, Theorem 3.2.2 is true in this generality; this is due to Colmez, see [Sc2, Proposition 4.8].

Theorem 3.2.2 is a remarkable assertion: it allows us to reduce statements about (pro-)étale sheaves on rigid-analytic spaces to those for perfectoid spaces. In practice, this means that affinoid perfectoids play a role in p-adic geometry that is somewhat analogous to the role of unit polydisks in complex analytic geometry. We do not prove Theorem 3.2.2 in these notes. Instead, we content ourselves by describing the key construction that goes into its proof, which is analogous to the one in §2.2.

Example 3.2.4 (The perfectoid torus)**.** Let $X := \mathbb{T}^1 := \mathrm{Spa}(C\langle T^{\pm 1}\rangle, \mathcal{O}_C\langle T^{\pm 1}\rangle)$ be the torus. Consider the object $U := \{U_i\}_{i \in \mathbf{N}} \in X_{proet}$ given by setting $U_n = X$ for all n, with the transition map $U_{n+1} \to U_n$ being given by the p-power map on the torus. To avoid confusion, choose co-ordinates so as to write $U_n = \mathrm{Spa}(C\langle T^{\pm \frac{1}{p^n}}\rangle, \mathcal{O}_C\langle T^{\pm \frac{1}{p^n}}\rangle)$. Then U is indeed affinoid perfectoid: the corresponding perfectoid affinoid algebra is simply $(C\langle T^{\pm \frac{1}{p^\infty}}\rangle, \mathcal{O}_C\langle T^{\pm \frac{1}{p^\infty}}\rangle)$. Note that each map $U_{n+1} \to U_n$ is a $\mu_p(C)$-torsor, and hence $U \to X$ is a pro-étale $\mathbf{Z}_p(1)$-torsor. Explicitly, we have a (continuous) direct sum decomposition

$$C\langle T^{\pm \frac{1}{p^\infty}}\rangle \simeq \widehat{\bigoplus_{i \in \mathbf{Z}[\frac{1}{p}]}} C \cdot T^i.$$

[3] The definition of the pro-étale site has evolved a bit over time. For our purposes, the original one from [Sc2, §3], which is perhaps the most intuitive, suffices. Other variants that are technically much more useful were discovered later, and are discussed in the lectures of Kedlaya and Weinstein. In particular, the notion discussed in these notes is called the *flattening pro-étale topology* in Kedlaya's lectures. The reader may freely use any of these variants whilst reading these notes.

This decomposition is equivariant for the $\mathbf{Z}_p(1)$-action, and an element $\underline{\epsilon} := (\epsilon_n) \in \lim \mu_{p^n}(C) =: \mathbf{Z}_p(1)$ acts on the summands via

$$T^{\frac{a}{p^m}} \mapsto \epsilon_m^a T^{\frac{a}{p^m}}.$$

For convenience, we often abbreviate this action as

$$T^i \mapsto \underline{\epsilon}^i T^i.$$

In particular, in this case, we have a profinite étale cover of X by an affinoid perfectoid in X_{proet}. More generally, a similar construction applies when X admits an étale map to the n-dimensional torus $\mathbb{T}^n$ that factors as a composition of rational subsets and finite étale maps (see [Sc2, Lemma 4.6]). In general, one can always cover X by affinoid opens that admit such maps.

We now recall some "vanishing theorems" on X_{proet}. Recall that we have already discussed the morphism $\nu : X_{proet} \to X_{et}$ of sites. Using this morphism, we obtain the sheaves $\mathcal{O}_X^+ := \nu^*\mathcal{O}_{X_{et}}^+$ and $\mathcal{O}_X := \nu^*\mathcal{O}_{X_{et}}$ on X_{proet}; here $\mathcal{O}_{X_{et}}$ and $\mathcal{O}_{X_{et}}^+$ are the usual structure sheaves on the étale site X_{et}. The completed structure sheaves are then defined as $\widehat{\mathcal{O}_X^+} = \lim \mathcal{O}_X^+/p^n$ and $\widehat{\mathcal{O}_X} = \widehat{\mathcal{O}_X^+}[\frac{1}{p}]$. Given an affinoid perfectoid $U := \{\mathrm{Spa}(R_i, R_i^+)\} \in X_{proet}$ as in Definition 3.2.1 with limit $\widehat{U} := \mathrm{Spa}(R, R^+)$, one has the expected formulae

$$\widehat{\mathcal{O}_X^+}(U) = R^+ \quad \text{and} \quad \widehat{\mathcal{O}_X}(U) = R,$$

see [Sc2, Lemma 4.10]. The first vanishing theorem concerns the cotangent complex:

Corollary 3.2.5. *The cotangent complex $L_{\widehat{\mathcal{O}_X^+}/\mathcal{O}_C}$ vanishes modulo p on X_{proet}. Hence, the p-adic completion of $L_{\widehat{\mathcal{O}_X^+}/\mathcal{O}_C}$ vanishes.*

Proof. By Theorem 3.2.2, it is enough to show that the presheaf $U \mapsto L_{\widehat{\mathcal{O}_X^+}(U)/\mathcal{O}_C} \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p$ vanishes on affinoid perfectoid $U \in X_{proet}$. But $\widehat{\mathcal{O}_X^+}(U) = R^+$ for a perfectoid affinoid (R, R^+). We are then reduced to the vanishing modulo p of the cotangent complex for perfectoids, which may be deduced from Proposition 3.1.6 as $\mathcal{O}_C/p \to R^+/p$ is flat and relatively perfect. □

In other words, there is no differential geometric information available when working on the ringed site $(X_{proet}, \widehat{\mathcal{O}_X})$. We shall see later that the differential forms on X can nevertheless be recovered from $(X_{proet}, \widehat{\mathcal{O}_X})$ via pushforward down to X_{et}. The second vanishing theorem concerns the cohomology of $\widehat{\mathcal{O}_X}$ on affinoid perfectoids (see [Sc2, Lemma 4.10]):

Theorem 3.2.6 (Acyclicity of the structure sheaf on affinoid perfectoids)**.** *Let $U \in X_{proet}$ be an affinoid perfectoid. Then $H^i(U, \widehat{\mathcal{O}_X^+})$ is almost zero[4] for $i > 0$, and thus $H^i(U, \widehat{\mathcal{O}_X}) = 0$ for $i > 0$.*

[4] The phrase "almost zero" refers to a notion introduced by Faltings [Fa1]: an $\mathcal{O}_C$-module is almost zero if it is killed by the maximal ideal of $\mathcal{O}_C$. Intuitively, such a module is "very small" and can often be safely ignored when performing computations. Faltings theory of "almost mathematics" (expounded in [GR]) is based on the idea of systematically developing various notions of commutative algebra and algebraic geometry up to almost zero error terms (as in Theorem 3.2.6), i.e., one works with rings, modules, etc. in the $\otimes$-category of almost $\mathcal{O}_C$-modules, defined as the quotient of the category of all $\mathcal{O}_C$-modules by almost zero ones. Whilst we have avoided any discussion of this notion in these notes, it is important to note that almost mathematics lurks in the background when working with perfectoid spaces, is most directly visible in the integral aspects of theory.

In particular, this theorem gives us a technique for calculating the cohomology of $\widehat{\mathcal{O}_X}$ for any affinoid $U \in X_{proet}$: if we choose a pro-étale cover $V \to U$ with V affinoid perfectoid as provided by Theorem 3.2.2, then Cech theory gives an identification

$$H^i(U, \widehat{\mathcal{O}_X}) \simeq H^i\Big(\widehat{\mathcal{O}_X}(V) \to \widehat{\mathcal{O}_X}(V \times_U V) \to \widehat{\mathcal{O}_X}(V \times_U V \times_U V) \to ...\Big)$$

as V, $V \times_U V$, $V \times_U V \times_U V$, etc. are all affinoid perfectoid; here the differentials are the alternating sums of the pullbacks along the various projections. If we can further ensure that $V \to U$ is a G-torsor for a profinite group (see Example 3.2.4 for an example), then $V \times_U V \simeq V \times \underline{G}$ (where $\underline{G}$ is the "topologically constant" sheaf on X_{proet} defined by the association $W \mapsto \mathrm{Map}_{conts}(|W|, G)$, where $|W|$ is the natural topological space attached to $W \in X_{proet}$), so the above formula simplifies to

$$H^i(U, \widehat{\mathcal{O}_X}) = H^i_{conts}(G, \widehat{\mathcal{O}_X}(V)).$$

In other words, we can calculate the cohomology of $\widehat{\mathcal{O}_X}$ in terms of the continuous group cohomology. The same strategy also applies for the integral sheaf $\widehat{\mathcal{O}_X^+}$ in the almost category, and will be used repeatedly in the sequel.

3.3 The key calculation

Continuing the notation from §3.2, we record the main calculation describing $R\nu_*\widehat{\mathcal{O}_X}$.

Lemma 3.3.1. *The $\mathcal{O}_X$-module $R^1\nu_*\widehat{\mathcal{O}_X}$ is locally free of rank n, and taking cup products gives an isomorphism $\wedge^i R^1\nu_*\widehat{\mathcal{O}_X} \simeq R^i\nu_*\widehat{\mathcal{O}_X}$.*

Proof. This is a local assertion, so we may assume that X is affinoid, and that there exists an étale map $X \to \mathbb{T}^n$ that factors as a composition of rational subsets and finite étale covers. By the vanishing of higher coherent sheaf cohomology on affinoids, it is enough to show the following:

1. The $\mathcal{O}_X(X)$-module $H^1(X_{proet}, \widehat{\mathcal{O}_X})$ is free of rank n.
2. Taking cup products gives an isomorphism $\wedge^i H^1(X_{proet}, \widehat{\mathcal{O}_X}) \simeq H^i(X_{proet}, \widehat{\mathcal{O}_X})$ for each i.
3. The preceding two properties are compatible with étale localization on X.

We shall explain the first two in the key example of a torus, leaving the rest to the references.

Consider first the case $X = \mathbb{T}^1 := \mathrm{Spa}(C\langle T^{\pm 1}\rangle, \mathcal{O}_C\langle T^{\pm 1}\rangle)$ of a 1-dimensional torus with co-ordinate T. Write $X_\infty \in X_{proet}$ for the affinoid perfectoid object constructed in Example 3.2.4. Then Theorem 3.2.6 shows that

$$R\Gamma(X_{\infty,proet}, \widehat{\mathcal{O}_X}) \simeq C\langle T^{\pm \frac{1}{p^\infty}}\rangle.$$

As $X_\infty \to X$ is a $\mathbf{Z}_p(1)$-torsor, this implies (see discussion following Theorem 3.2.6) that

$$R\Gamma(X_{proet}, \widehat{\mathcal{O}_X}) \simeq R\Gamma_{conts}(\mathbf{Z}_p(1), C\langle T^{\pm \frac{1}{p^\infty}}\rangle).$$

Now that canonical presentation

$$C\langle T^{\pm\frac{1}{p^\infty}}\rangle \simeq \widehat{\bigoplus_{i\in\mathbf{Z}[\frac{1}{p}]}} C\cdot T^i$$

is equivariant for the action of $\mathbf{Z}_p(1)$ described in Example 3.2.4. In particular, if $\underline{\epsilon} = (\epsilon_n) \in \lim \mu_{p^n}(C) = \mathbf{Z}_p(1)$ is a generator, then, by standard facts about the continuous group cohomology of pro-cyclic groups, we have

$$R\Gamma(X_{proet}, \widehat{\mathcal{O}_X}) \simeq \widehat{\bigoplus_{i\in\mathbf{Z}[\frac{1}{p}]}} \left(C\cdot T^i \xrightarrow{T^i\mapsto(\underline{\epsilon}^i-1)T^i} C\cdot T^i\right);$$

here we follow the convention that if $i = \frac{a}{p^m}$ $a \in \mathbf{Z}$, then $\underline{\epsilon}^i = \epsilon_m^a$. In particular, the differential is trivial on the summands indexed by $i \in \mathbf{Z}$ (as $\underline{\epsilon}^i = 1$ for such i) and an isomorphism for non-integral $i \in \mathbf{Z}[\frac{1}{p}]$ (as $\underline{\epsilon}^i - 1 \neq 0$ for such i). Thus, up to quasi-isomorphism, we can ignore the non-integral summands to get

$$R\Gamma(X_{proet}, \widehat{\mathcal{O}_X}) \simeq \widehat{\bigoplus_{i\in\mathbf{Z}}}\left(C\cdot T^i \xrightarrow{0} C\cdot T^i\right).$$

This presentation (and some unraveling of isomorphisms) shows that $H^*(X_{proet}, \widehat{\mathcal{O}_X})$ is the exterior algebra on its H^1, and that $H^1(X_{proet}, \widehat{\mathcal{O}_X})$ is free of rank 1, as wanted.

The preceding analysis applies equally well (modulo bookkeeping) when $X = \mathbb{T}^n$ is an n-dimensional torus for any $n \geq 1$. The general case is then deduced from this one by the almost purity theorem and base change properties of group cohomology, as explained in [Sc3, Proposition 3.23] and [Sc2, Lemma 4.5, 5.5]. □

3.4 Construction of the map

In this section, we give a global construction of the map

$$\Phi^i : \Omega^i_{X/C}(-i) \to R^i\nu_*\widehat{\mathcal{O}_X}$$

that will eventually give the isomorphism in Theorem 3.0.3. This construction is analogous to the one in [BMS2, §8.2] and differs from that in [Sc3, §3.3].

We choose a formal model $\mathfrak{X}/\mathcal{O}_C$ of X, and write $\mathfrak{X}_{aff}$ for the category of affine opens in $\mathfrak{X}$ with the indiscrete topology (so all presheaves are sheaves). Then we have evident morphisms

$$(X_{proet}, \widehat{\mathcal{O}_X}) \xrightarrow{\nu} (X_{et}, \mathcal{O}_X) \xrightarrow{\pi} (\mathfrak{X}_{aff}, \mathcal{O}_{\mathfrak{X}})$$

of ringed sites, and write $\mu = \pi \circ \nu$ for the composite. We shall construct[5] a natural morphism

$$\Phi^{1,'} : \Omega^1_{\mathfrak{X}/\mathcal{O}_C} \to R^1\mu_*\widehat{\mathcal{O}_X}(1).$$

[5]Here $\Omega^1_{\mathfrak{X}/\mathcal{O}_C}$ denotes the sheaf of Kähler differentials on the formal scheme, and is computed as follows: if $\mathfrak{X} = \mathrm{Spf}(R)$ for flat $\mathcal{O}_C$-algebra R that is topologically of finite presentation, then $\Omega^1_{\mathfrak{X}/\mathcal{O}_C}$ is the coherent $\mathcal{O}_{\mathfrak{X}}$-sheaf associated to the finitely presented R-module of continuous Kähler differentials on R, see [EGA, §0.20.1] and [GR, 7.1.23]. This module is computed as the p-adic completion of module $\Omega^1_{R/\mathcal{O}_C}$ in the algebraic sense. In particular, the sheaf $\Omega^1_{\mathfrak{X}/\mathcal{O}_C}$ has the following key feature for our purposes: its values on affines are p-adically complete.

By formal properties of adjoints, this defines a map

$$\pi^*\Omega^1_{\mathfrak{X}/\mathcal{O}_C} = \pi^{-1}\Omega^1_{\mathfrak{X}/\mathcal{O}_C} \otimes_{\pi^{-1}\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_X \to R^1\nu_*\widehat{\mathcal{O}_X}(1).$$

The left side identifies with $\Omega^1_{X/C}$, so untwisting defines the desired map Φ^1. The remaining Φ^i's are obtained by passage to exterior powers using the anticommutative cup product on $\oplus_i R^i\nu_*\widehat{\mathcal{O}_X}$.

Consider the maps

$$\mathbf{Z}_p \to \mathcal{O}_C \to \widehat{\mathcal{O}_X^+}$$

of sheaves of rings on X_{proet}. Attached to this, there is a standard exact triangle

$$L_{\mathcal{O}_C/\mathbf{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}_X^+} \to L_{\widehat{\mathcal{O}_X^+}/\mathbf{Z}_p} \to L_{\widehat{\mathcal{O}_X^+}/\mathcal{O}_C}$$

of cotangent complexes. Corollary 3.2.5 shows that the last term vanishes after a p-adic completion. Hence, we obtain an isomorphism

$$\widehat{L_{\mathcal{O}_C/\mathbf{Z}_p} \otimes_{\mathcal{O}_C}} \widehat{\mathcal{O}_X^+} \simeq \widehat{L_{\widehat{\mathcal{O}_X^+}/\mathbf{Z}_p}}.$$

By Theorem 2.3.1, the first term identifies with $\Omega \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}_X^+}[1]$, where Ω is a free $\mathcal{O}_C$-module of rank 1 that Galois equivariantly looks like $\mathcal{O}_C(1)$ up to torsion. In particular, inverting p gives

$$\widehat{\mathcal{O}_X}(1)[1] \simeq \widehat{L_{\widehat{\mathcal{O}_X^+}/\mathbf{Z}_p}}[\frac{1}{p}]. \tag{3.1}$$

Now consider the map μ of ringed sites. Via pullback, this yields a map

$$\widehat{L_{\mathfrak{X}/\mathbf{Z}_p}} \to R\mu_*\widehat{L_{\widehat{\mathcal{O}_X^+}/\mathbf{Z}_p}} \to R\mu_*\widehat{L_{\widehat{\mathcal{O}_X^+}/\mathbf{Z}_p}}[\frac{1}{p}] \simeq R\mu_*\widehat{\mathcal{O}_X}(1)[1]. \tag{3.2}$$

To proceed further, we claim that there is a natural identification

$$\mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathbf{Z}_p}}) \simeq \Omega^1_{\mathfrak{X}/\mathcal{O}_C}. \tag{3.3}$$

Granting this claim, passage to $\mathcal{H}^0$ in (3.2) yields the map

$$\Phi^{1,'} : \Omega^1_{\mathfrak{X}/\mathcal{O}_C} \to R^1\mu_*\widehat{\mathcal{O}_X}(1),$$

and hence the maps Φ^i, as explained earlier. To prove (3.4), consider the sequence

$$\mathbf{Z}_p \to \mathcal{O}_C \to \mathcal{O}_{\mathfrak{X}}$$

of rings on $\mathfrak{X}_{aff}$. The transitivity triangle then takes the form

$$L_{\mathcal{O}_C/\mathbf{Z}_p} \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X}} \to L_{\mathfrak{X}/\mathbf{Z}_p} \to L_{\mathfrak{X}/\mathcal{O}_C}.$$

On applying the derived p-adic completion functor, we obtain an exact triangle where the term on the left has no $\mathcal{H}^0$, so we obtain an identification

$$\mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathbf{Z}_p}}) \simeq \mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathcal{O}_C}}).$$

The upshot of this reduction is that the right side is a geometric object: $\mathfrak{X}$ is a topologically finitely presented formal scheme over $\mathcal{O}_C$. Using this fact, one can check that

$$\mathcal{H}^0(\widehat{L_{\mathfrak{X}/\mathcal{O}_C}}) = \Omega^1_{\mathfrak{X}/\mathcal{O}_C},$$

which then gives the desired (3.4); we refer to the discussion surrounding [GR, Lemma 7.1.25] for more on the relationship between the cotangent complex and continuous Kähler differentials, and [GR, Proposition 7.1.27] for the proof of the above equality.

3.5 Conclusion: the isomorphy of Φ^i

Combining the material in §3.4 with the calculation §3.3, we learn that both the source and the target of

$$\oplus_i \Phi^i : \bigoplus_i \wedge^i \big(\Omega^1_{X/C}(-1)\big) \to \bigoplus_i R^i \nu_* \widehat{\mathcal{O}_X}$$

are exterior algebras on the $i = 1$ terms. Thus, to prove that Φ^i is an isomorphism for all i, it suffices to do so for $i = 1$. Moreover, note that both sides are coherent sheaves of an étale local nature on X. Thus, we may assume $X = \mathbb{T}^n$, and may pass to global sections. Thus, we need to show that the

$$\Phi^1(X) : \Omega^1_{X/C}(-1) \to H^1(X_{proet}, \widehat{\mathcal{O}_X})$$

of free rank n $\mathcal{O}_X(X)$-modules is an isomorphism. Both sides are compatible with taking products of adic spaces, so one reduces to the case $n = 1$. Choose coordinates to write $X := \mathbb{T}^1 := \mathrm{Spa}(C\langle T^{\pm 1}\rangle, \mathcal{O}_C\langle T^{\pm 1}\rangle)$. Then $d\log(T) \in \Omega^1_{X/C}$ is a generator, and it suffices to show that $\Psi^1(d\log(T))$ is also a generator. This can be checked by making explicit the construction of §3.4 as in [BMS2, §8.3].

4. Lecture 4: Integral aspects

Let $C/\mathbf{Q}_p$ be a complete and algebraically closed field with residue field k. Let $\mathfrak{X}$ be a smooth and proper formal scheme[1] over $\mathcal{O}_C$. Write $X = \mathfrak{X}_C$ for the generic fibre, and $\mathfrak{X}_k$ for the special fibre. We then have the degenerate Hodge-Tate spectral sequence

$$E_2^{i,j} : H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X, C).$$

leading to the equality

$$\dim_{\mathbf{Q}_p} H^n(X_{et}, \mathbf{Q}_p) = \dim_C H^n(X, C) = \sum_{i+j=n} \dim_C H^i(X, \Omega^j_{X/C}) \tag{4.1}$$

relating étale and Hodge-cohomology for the generic fiber. As X admits a good model $\mathfrak{X}$, the groups appearing on either side of the equality above admit good integral and mod-p variants: we have

$$H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}/k}) \quad \text{and} \quad H^n(X_{et}, \mathbf{F}_p).$$

It is thus natural to ask if (4.1) admits a mod-p variant. The following theorem was proven recently in [BMS2]

Theorem 4.0.1. *One has inequalities*

$$\dim_{\mathbf{F}_p} H^n(X_{et}, \mathbf{F}_p) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}/k}). \tag{4.2}$$

4.1 Examples

In this section, we record some examples showing that the inequality in Theorem 4.0.1 can be strict. The strategy is to construct certain interesting degenerations of group schemes, and then to approximate their classifying stacks. To motivate this idea and subsequent constructions, we begin with a purely topological calculation.

[1]Not much will be lost if one assumes that $C = \mathbf{C}_p$ and that $\mathfrak{X}$ arises as the p-adic completion of a proper smooth $\mathcal{O}_C$-scheme $\mathcal{X}$. For our constructions, though, it will nevertheless be convenient to work with formal schemes. The added generality is also useful in some geometric applications, see [CLL] for a recent concrete example arising from the following phenomenon: even though K3 surfaces over C with good reduction might only do so in the world of algebraic spaces, the special fibre of a good model will be a scheme, and hence formal completion of a good model will be a formal scheme.

Example 4.1.1. Let $G = \mathbf{Z}/p$. Consider the classifying space BG of G-torsors; this space can be defined as EG/G, where EG is a contractible space with a free G-action. The cohomology of BG agrees with the group cohomology of G. We claim that there exist G-torsors $f_i : X_i \to BG$ for $i \in \{0,1\}$ such that $H^1(X_0, \mathbf{F}_p) \simeq 0$, but $H^1(X_1, \mathbf{F}_p) \neq 0$. In fact, for X_0, we take $X_0 = EG$, with $f_0 : X_0 \to BG$ being the universal G-torsor: as EG is contractible, we have $H^{>0}(X_0, \mathbf{F}_p) \simeq 0$. For X_1, we simply take $X_1 = BG \times G$ with $f_1 : X_1 \to BG$ being the projection, realizing X_1 as the trivial G-torsor over BG. Then $H^1(X_1, \mathbf{F}_p)$ contains $H^1(BG, \mathbf{F}_p)$ as a summand, and is thus nonzero since $H^1(BG, \mathbf{F}_p) \simeq \mathrm{Hom}(G, \mathbf{F}_p) \neq 0$.

As a thought experiment, imagine that one can construct a family degenerating f_0 to f_1, i.e., a continuous one parameter family $f_t : X_t \to BG$ of G-torsors indexed by $t \in [0,1]$ coinciding with the construction above for $t = 0, 1$. The total space $\mathcal{X}$ of this degeneration would then admit a fibration $\mathcal{X} \to [0,1]$ whose fibers have varying $\mathbf{F}_p$-cohomologies. Unfortunately, it is impossible to find such a degeneration in topology. Indeed, any such family would correspond to a non-constant path in the "space of G-torsors on BG" that degenerates the non-trivial torsor X_0 to the trivial torsor X_1. The space of such torsors is tautologically $\mathrm{Map}(BG, BG)$; as G is discrete and abelian, this space admits no non-trivial paths[2], so no such families exist. (Even more directly, the fibers of a fibration over $[0,1]$ are homotopy-equivalent, and hence can't have distinct cohomologies.)

However, we *can* produce such a degeneration in algebraic geometry in positive or mixed characteristic, essentially because morphisms between finite group schemes can vary in families in this setting; for example, $\mathrm{Hom}(\mathbf{Z}/p, \mu_p) \simeq \mu_p$ is not discrete in characteristic p. Using this idea, one can rather readily find the phenomenon described in the previous paragraph in the world of algebraic stacks (see [BMS1, Example 4.1]). To stay within the world of schemes, one needs an additional approximation argument. The example recorded next (from [BMS2, §2.1]) accomplishes both of these tasks, albeit in a hidden fashion.

Example 4.1.2. Assume $p = 2$. Let $S/\mathcal{O}_C$ be a proper smooth morphism with $\pi_1(S_C) \xrightarrow{\simeq} \pi_1(S) \xleftarrow{\simeq} \pi_1(S_k) \simeq \mathbf{Z}/2$; one may construct an Enriques surface with such properties. Let $E/\mathcal{O}_C$ be an elliptic curve with good ordinary reduction. Hence, there is a canonical subgroup $\mu_2 \subset E$ (see Caraiani's lectures). Choosing the element $-1 \in \mu_2(\mathcal{O}_C)$ defines a map

$$\alpha : \underline{\mathbf{Z}/2} \to \mu_2 \subset E$$

of group schemes over $\mathcal{O}_C$. If $\tilde{S} \to S$ denotes the universal $\mathbf{Z}/2$-cover of S, then we may push out $\tilde{S} \to S$ along α to obtain an E-torsor $f : Y \to S$ by setting $Y := \tilde{S} \times_{\mathbf{Z}/2} E$ (where $\mathbf{Z}/2$ acts via the covering involution on $\tilde{S}$, and by translation using α on E) with $f : Y \to S$ the map induced by projection onto $\tilde{S}/(\mathbf{Z}/2) \simeq S$. This E-torsor has the following properties:

1. The special fiber $Y_k \to S_k$ is identified with the split torsor $E_k \times S_k \to S_k$: the construction of Y is compatible with restriction to the special fibre, and α_k is the 0 map as $-1 = 1$ over k.

[2] More precisely, for any pair of discrete groups H and G, the space $\mathrm{Map}(BH, BG)$ can be modeled by a groupoid whose objects are group homomorphisms $f : H \to G$, and morphisms $f \to f'$ are given by group elements $g \in G$ that conjugate f to f'. When G is abelian (as above), this description collapses to identify $\mathrm{Map}(BH, BG)$ as the product of the discrete set $\mathrm{Hom}(H, G)$ with the groupoid BG. In particular, a path in $[0,1] \to \mathrm{Map}(BH, BG)$ is "trivial," i.e., the corresponding map $BH \times [0,1] \to BG$ factors through the second projection.

2. The generic fibre $Y_C \to S_C$ is a non-split E_C-torsor (i.e., it has no section): using the exact sequence

$$0 \to \underline{\mathbf{Z}/2}_C \xrightarrow{\alpha_C} E_C \xrightarrow{\beta} E'_C \to 0 \tag{4.3}$$

(where E'_C is defined as the quotient of the elliptic curve E_C by the non-trivial 2-torsion point coming from α, and is thus also an elliptic curve over C), the triviality of the torsor $Y_C \to S_C$ would give a non-constant map $S_C \to E'_C$. But one can show that there are no non-constant maps from a smooth proper variety over C with finite étale fundamental group into an abelian variety[3], so we are done.

We now calculate both sides of (4.2) in this example in degree 1. On the étale side, we give a topological argument after choosing an isomorphism $C \simeq \mathbf{C}$; alternately, a purely algebraic version of the same set of ideas can be found in [BMS2, §2.1]. The map $Y_C \to S_C$ is an E_C-torsor, so fixing a (suppressed) base point on S_C gives an exact sequence of homotopy groups

$$\pi_1(E_C) \xrightarrow{\mu} \pi_1(Y_C) \xrightarrow{\nu} \pi_1(S_C) \to 0,$$

where the surjectivity on the right comes from the connectedness of the fibers. We shall show that μ is injective, and identify the resulting extension. Consider the map $Y \to E'_C := E_C/(\mathbf{Z}/2)$ coming from the definition of Y. The composite $E_C \to Y \to E'_C$ is clearly injective on π_1 (as it is a non-constant map of smooth proper curves of genus 1), and thus μ must be injective. This data fits into a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(E_C) & \xrightarrow{\mu} & \pi_1(Y_C) & \xrightarrow{\nu} & \pi_1(S_C) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \eta} & & \\ 0 & \longrightarrow & \pi_1(E_C) & \longrightarrow & \pi_1(E'_C) & \xrightarrow{\tau} & \mathbf{Z}/2 & \longrightarrow & 0. \end{array}$$

Here the target of τ is identified via the boundary map induced by the fibration coming from the short exact sequence (4.3). Unraveling definitions shows that η is the identity; in fact, slightly more canonically, the target of τ is naturally $\mu_2(C)$ viewed as the canonical subgroup on $E(C)$, and the map η arises from our choice of $-1 \in \mu_2(\mathcal{O}_C)$ at the start of the construction defining α. Putting these together, we see that η is an isomorphism, and hence $\pi_1(Y_C) \simeq \pi_1(E'_C) \simeq \mathbf{Z}^{\oplus 2}$. In particular, we get

$$\dim_{\mathbf{F}_2} H^1(Y_{C,et}, \mathbf{F}_2) = 2. \tag{4.4}$$

We now move to the Hodge side. Here, we have $Y_k \simeq S_k \times E_k$. In particular, one has

$$h^{0,1}(Y_k) = h^{0,1}(S_k) + h^{0,1}(E_k) \quad \text{and} \quad h^{1,0}(Y_k) = h^{1,0}(S_k) + h^{1,0}(E_k)$$

by the Künneth formula for the cohomology of the structure sheaf and differential forms. Now $h^{0,1}(E_k) = h^{1,0}(E_k) = 1$ by general facts about elliptic curves. Also, we claim that $H^1(S_k, \mathcal{O}_{S_k}) \neq 0$, and hence

[3]Fix a map $g : Z \to A$ over C, where Z is a smooth proper variety, A is an abelian variety, and $\pi_1^{et}(Z)$ is finite. Then the induced map $g_* : \pi_1^{et}(Z) \to \pi_1^{et}(A)$ is constant as $\pi_1^{et}(A)$ is topologically a free abelian group, and thus the pullback $g^* : H^1(A, C) \to H^1(Z, C)$ is the 0 map. As $H^*(A, C) \simeq \wedge^* H^1(A, C)$ via cup products, it follows that $g^* : H^n(A, C) \to H^n(Z, C)$ is 0 for all $n > 0$. In particular, if $L \in \mathrm{Pic}(A)$ is an ample line bundle, then $c_1(g^*L) = g^*c_1(L)$ is 0. On the other hand, if g was non-constant, then there would exist a curve $i : C \hookrightarrow Z$ such that $g \circ i : C \to A$ is finite, and thus i^*g^*L is ample, so $\deg(i^*g^*L) = c_1(i^*g^*L) = i^*c_1(g^*L)$ is positive. This contradicts the triviality of $c_1(g^*L)$, so there are no such curves, and hence g must be constant.

$h^{0,1}(S_k) > 0$: as $\pi_1(S_k) \simeq \mathbf{Z}/2$, there is a non-trivial element in $H^1(S_{k,et}, \mathbf{F}_2)$, which contributes a non-trivial element to $H^1(S_k, \mathcal{O}_{S_k})$ from the Artin-Schreier exact sequence

$$0 \to \mathbf{F}_2 \to \mathcal{O}_{S_k} \xrightarrow{F-1} \mathcal{O}_{S_k} \to 0.$$

Putting these together, we learn that

$$\dim_k H^1(Y_k, \mathcal{O}_{Y_k}) + \dim_k H^0(Y_k, \Omega^1_{Y_k/k}) \geq 3. \tag{4.5}$$

Comparing (4.4) and (4.5) shows that (4.2) can be strict.

The inequality (4.2) is a consequence of the following stronger inequality

$$\dim_{\mathbf{F}_p} H^n(X_{et}, \mathbf{F}_p) \leq \dim_k H^n_{dR}(\mathfrak{X}_k/k),$$

proven in (4.6) below. Now both sides have a canonical mixed characteristic deformation: étale cohomology with $\mathbf{Z}_p$-coefficients on the left, and crystalline cohomology on the right. In fact, as explained in (4.7), the previous inequality may be improved to compare the torsion in the two lifts: one has

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}) \leq \ell_{W(k)}(H^i_{crys}(\mathfrak{X}_k/W(k))_{tors}).$$

It is natural to ask if this last inequality is actually a reflection of an inclusion of groups. For example, if $H^i(X_{et}, \mathbf{Z}_p)$ contains an element of order p^2, is the same true for $H^i_{crys}(\mathfrak{X}_k/W(k))$? We shall answer this question negatively. The crucial idea going into the construction of the example is again a phenomenon exhibited by finite flat group schemes away from equicharacteristic 0: one can degenerate a finite group scheme of order exactly p^2 in characteristic 0 into a finite group scheme that is killed by p in characteristic p. An explicit construction of such a degeneration is recorded next.

Construction 4.1.3. Let $\mathcal{E}/\mathcal{O}_C$ be an elliptic curve with supersingular reduction. Choose a point $x \in \mathcal{E}(C)$ of order exactly p^2, so x defines an inclusion $\underline{\mathbf{Z}/p^2} \hookrightarrow \mathcal{E}_C$ of group schemes. Taking the closure, we obtain a finite flat group subscheme $G \hookrightarrow \mathcal{E}$ with $G|_C \simeq \underline{\mathbf{Z}/p^2}$. The special fibre $G_k \subset \mathcal{E}_k$ is a subgroup of order p^2 on the elliptic curve $\mathcal{E}_k$. As $\mathcal{E}_k$ is supersingular, all p-power torsion subgroups of $\mathcal{E}_k$ are infinitesimal. In particular, there is a unique subgroup of order p^2, given (as a scheme) by the (p^2-1)-th infinitesimal neighbourhood of $0 \in \mathcal{E}_k$. As $\mathcal{E}_k[p]$ is a subgroup of order p^2, we must have $G_k = \mathcal{E}_k[p]$, so G_k is killed by p, while G_C is a cyclic group scheme of exact order p^2.

Passing from the above construction of group schemes to their classifying stacks yields the sought-for examples in the world of algebraic stacks; the example below approximates this construction using smooth projective varieties.

Example 4.1.4. Choose G as in Construction 4.1.3. Then we may choose a smooth projective $\mathcal{O}_C$-scheme $\mathfrak{Y}$ that has relative dimension 2 and comes equipped with a free G-action. In fact, one may (and we do) choose[4] $\mathfrak{Y}$ to be a general complete intersection surface in $\mathbf{P}^n$ for $n \gg 0$. Set $\mathfrak{X} = \mathfrak{Y}/G$ to be the quotient, so $\mathfrak{X}$ is a smooth projective $\mathcal{O}_C$-scheme of relative dimension 2 equipped with a G-torsor $\pi : \mathfrak{Y} \to \mathfrak{X}$.

[4]The existence of such complete intersections is a general fact that is valid for all finite flat group schemes; this fact goes back to the work of Serre [Se] and Atiyah-Hirzeburch [AH]. More recent accounts of this construction include [To, §1], [MV, §4.2], and [Ill3, §6], and the details necessary for our purposes can be found in [BMS2, §2.2].

On the étale side, the Hochschild-Serre spectral sequence for the G-torsor π shows that $H^2(X_{C,et}, \mathbf{Z}_p)_{tors} \simeq \mathbf{Z}/p^2$. Indeed, as $\mathcal{Y}$ is a complete intersection surface, the groups $H^i(\mathcal{Y}_{C,et}, \mathbf{Z}_p)$ are torsionfree for $i \in \{0, 2\}$ and 0 for $i = 1$ by the Lefschetz theorems; the desired claim immediately falls out of the low degree terms for the spectral sequence. Slightly more conceptually, the G-torsor π is classified by a map $\mathfrak{X} \to BG$; we have $H^2(BG_{C,et}, \mathbf{Z}_p)_{tors} = H^2(\mathbf{Z}/p^2, \mathbf{Z}_p)_{tors} = \mathbf{Z}/p^2$, and this group maps isomorphically to $H^2(X_{et}, \mathbf{Z}_p)_{tors}$.

On the crystalline side, we claim that $H^2_{crys}(\mathfrak{X}_k/W(k))_{tors}$ is killed by multiplication by p. By repeating the reasoning used above, we are reduced to showing that $H^i_{crys}(BG_k/W(k))$ is killed by multiplication by p. But G_k itself is killed by multiplication by p, and hence so is its cohomology. (The argument given in the last sentence is meant to convey intuition, and is not a rigorous one as the relevant technology to analyze the crystalline cohomology of stacks has not been documented (to the best of the author's knowledge); a more indirect but precise argument can be found in [BMS2, §2.2].)

Putting the conclusions of the previous paragraphs together, we learn that $H^2(X_{et}, \mathbf{Z}_p)_{tors}$ contains an element of order p^2, while $H^2_{crys}(\mathfrak{X}_k/W(k))_{tors}$ is killed by p. In particular, the length inequality

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}) \leq \ell_{W(k)}(H^i_{crys}(\mathfrak{X}_k/W(k))_{tors})$$

cannot be upgraded to an inclusion of groups.

4.2 The main theorem

Fix a complete and algebraically closed field $C/\mathbf{Q}_p$ with residue field k. As C is a perfectoid field, its valuation ring $\mathcal{O}_C$ is integral perfectoid, giving rise to its deformation $A_{inf} := A_{inf}(\mathcal{O}_C)$ as in Proposition 3.1.9; write $\phi : A_{inf} \to A_{inf}$ for the automorphism deduced by functoriality from Frobenius on $\mathcal{O}/p$, and write $\widetilde{\theta} := \theta \circ \phi^{-1} : A_{inf} \to \mathcal{O}_C$. Writing $C^\flat$ for the fraction field of $\mathcal{O}_C^\flat$, we also have the maps $A_{inf} \to W(C^\flat)$ and $A_{inf} \to W(k)$ arising from the functoriality of $W(-)$, and the map $A_{inf} \to \mathcal{O}_C^\flat$ arising by setting $p = 0$. The scheme $\mathrm{Spec}(A_{inf})$ together with the points and divisors arising from all these maps is depicted in Figure 4.1 (which is borrowed from [Bh]).

Fix a proper smooth formal scheme $\mathfrak{X}/\mathcal{O}_C$ with generic fibre X of dimension d. Theorem 4.0.1 asserts the existence of a numerical inequality between two mod-p cohomology theories: one is topological in nature and is attached to the generic fibre X, while the other is algebro-geometric and is attached to the special fibre $\mathfrak{X}_k$. This inequality is deduced by constructing a specialization from one cohomology theory to the other over the base A_{inf}, as follows:

Theorem 4.2.1 (The A_{inf}-cohomology theory). *There exists a functorial perfect complex $R\Gamma_A(\mathfrak{X}) \in D(A_{inf})$ together with a Frobenius action $\phi_{\mathfrak{X}} : \phi^* R\Gamma_A(\mathfrak{X}) \to R\Gamma_A(\mathfrak{X})$ that is an isomorphism outside the divisor $\mathrm{Spec}(\mathcal{O}_C) \stackrel{\widetilde{\theta}}{\hookrightarrow} \mathrm{Spec}(A_{inf})$ defined by $\widetilde{\theta}$. Moreover, one has the following comparison isomorphisms[5]:*

1. *Étale cohomology: there exists a canonical ϕ-equivariant identification*

$$R\Gamma_A(\mathfrak{X}) \otimes_{A_{inf}} W(C^\flat) \simeq R\Gamma(X_{et}, \mathbf{Z}_p) \otimes W(C^\flat).$$

[5] See Figure 4.1 for a depiction of the loci in $\mathrm{Spec}(A_{inf})$ where this comparison isomorphisms take place.

In fact, such an isomorphism already exists after base change to $A_{inf}[\frac{1}{\mu}]$, where $\mu \in A_{inf}$ is the element from Remark 3.1.12.

2. *de Rham cohomology: there exists a canonical isomorphism*

$$R\Gamma_A(\mathfrak{X}) \otimes^L_{A_{inf},\theta} \mathcal{O}_C \simeq R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_C).$$

3. *Hodge-Tate cohomology: there exists an E_2-spectral sequence*

$$E_2^{i,j} : H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/\mathcal{O}_C})\{-j\} \Rightarrow H^{i+j}(\widetilde{\theta}^* R\Gamma_A(\mathfrak{X})).$$

Here the twist $\{-j\}$ refers to the Breuil-Kisin twist from Remark 3.1.13.

4. *Crystalline cohomology of the special fibre: there exists a canonical ϕ-equivariant identification*

$$R\Gamma_A(\mathfrak{X}) \otimes^L_{A_{inf}} W(k) \simeq R\Gamma_{crys}(\mathfrak{X}_k/W(k)).$$

In fact, the properness assumption on $\mathfrak{X}$ is only necessary for Theorem 4.2.1 (1): the de Rham, Hodge-Tate and crystalline comparisons hold true for any smooth formal scheme $\mathfrak{X}$. Applications of Theorem 4.2.1 include the following:

1. *Recovering the Hodge-Tate decomposition.* The element $\mu \in A_{inf}$ is invertible at the generic point of the divisor $\mathrm{Spec}(\mathcal{O}_C) \overset{\widetilde{\theta}}{\hookrightarrow} \mathrm{Spec}(A_{inf})$ (marked as the Hodge-Tate specialization in Figure 4.1). Thus, the base change of $R\Gamma_A(\mathfrak{X})$ along $A_{inf} \xrightarrow{\widetilde{\theta}} \mathcal{O}_C \subset C$ is described by both Theorem 4.2.1 (1) and (3). Combining these gives the Hodge-Tate spectral sequence from Theorem 3.0.1.

2. *Recovering the inequality in Theorem 4.0.1.* Consider the perfect complex $K := R\Gamma_A(\mathfrak{X}) \otimes_{A_{inf}} \mathcal{O}_C^\flat$ over the valuation ring $\mathcal{O}_C^\flat$ (which is labelled as the modular specalization in Figure 4.1). By Theorem 4.2.1 (1), we have

$$K \otimes C^\flat \simeq R\Gamma(X_{et}, \mathbf{F}_p) \otimes C^\flat.$$

By Theorem 4.2.1 (2) or (3), we have

$$K \otimes k \simeq R\Gamma_{dR}(\mathfrak{X}_k/k).$$

By semicontinuity for the ranks of the cohomology groups of a perfect complex, we learn that

$$\dim_{\mathbf{F}_p} H^n(X_{et}, \mathbf{F}_p) \leq \dim_k H^n_{dR}(\mathfrak{X}_k/k). \tag{4.6}$$

On the other hand, the existence of the Hodge-to-de Rham spectral sequence shows that

$$\dim_k H^n_{dR}(\mathfrak{X}_k/k) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}).$$

Combining these, we obtain (4.2).

3. *Relating torsion in étale to crystalline or de Rham cohomology.* The reasoning used above can be upgraded to show the following inequality

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}/p^n) \leq \ell_{W(k)}(H^i_{crys}(\mathfrak{X}_k/W(k))_{tors}/p^n)$$

for all $n \geq 0$, and thus

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}) \leq \ell_{W(k)}(H^i_{crys}(\mathfrak{X}_k/W(k))_{tors}). \tag{4.7}$$

In particular, if $H^i_{crys}(\mathfrak{X}_k/W(k))$ is torsion free, so is $H^i(X_{et}, \mathbf{Z}_p)$. Once one defines a suitable normalized length[6] for finitely presented torsion $\mathcal{O}_C$-modules, the de Rham analogs of the previous two inequalities also hold true, as observed by Cesnavicus [Ce, Theorem 4.12]: one has

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}/p^n) \leq \ell_{\mathcal{O}_C}(H^i_{dR}(\mathfrak{X}/\mathcal{O}_C)_{tors}/p^n)$$

for all $n \geq 0$, and thus

$$\ell_{\mathbf{Z}_p}(H^i(X_{et}, \mathbf{Z}_p)_{tors}) \leq \ell_{\mathcal{O}_C}(H^i_{dR}(\mathfrak{X}/\mathcal{O}_C)_{tors}). \tag{4.8}$$

Example 4.1.4 shows that this inequalities cannot be upgraded to an inclusion of groups in general.

4. *The zero locus of $\phi_{\mathfrak{X}}$.* Theorem 4.2.1 asserts that the map $\phi_{\mathfrak{X}} : \phi^* R\Gamma_A(\mathfrak{X}) \to R\Gamma_A(\mathfrak{X})$ is an isomorphism outside the divisor $\mathrm{Spec}(\mathcal{O}_C) \overset{\widetilde{\theta}}{\hookrightarrow} \mathrm{Spec}(A_{inf})$ defined by $\widetilde{\theta}$. Specializing this picture along $A_{inf} \to W(k)$ and using the crystalline comparison recovers the Berthelot-Ogus theorem [BO1, Theorem 1.3] that $\phi_{\mathfrak{X}_k}$ is an isogeny on $R\Gamma_{crys}(\mathfrak{X}_k/W(k))$.

5. *The absolute crystalline comparison theorem.* Recall from Remark 3.1.16 that Fontaine's period ring A_{crys} is defined as the p-adic completion of the divided power envelope of the map $\theta : A_{inf} \to \mathcal{O}_C$; concretely, we choose a generator $\xi \in \ker(\theta)$ and define A_{crys} as the p-adic completion of $A_{inf}[\{\frac{\xi^n}{n!}\}_{n\geq 1}] \subset A_{inf}[\frac{1}{p}]$. The Frobenius automorphism ϕ of A_{inf} induces a Frobenius endomorphism ϕ of A_{crys}. More conceptually, the ring A_{crys} may be regarded as the absolute crystalline cohomology of $\mathrm{Spec}(\mathcal{O}_C/p)$, with ϕ corresponding to Frobenius. The image of the map $\mathrm{Spec}(A_{crys}) \to \mathrm{Spec}(A_{inf})$ is depicted in Figure 4.1.

   The absolute crystalline cohomology $R\Gamma_{crys}(\mathfrak{X}_{\mathcal{O}/p})$ of $\mathfrak{X}_{\mathcal{O}_C/p}$ is naturally an A_{crys}-complex. One may show that this A_{crys}-complex lifts the de Rham cohomology $R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_C)$ of $\mathfrak{X}$ along the map $A_{crys} \to \mathcal{O}_C$ arising from θ, and lifts the crystalline cohomology $R\Gamma_{crys}(\mathfrak{X}_k/W(k))$ along the map $A_{crys} \to W(k)$ factoring the canonical map $A_{inf} \to W(k)$. For this object, one has the following comparison isomorphism, which unifies and generalizes Theorem 4.2.1 (2) and (4): there exists a canonical ϕ-equivariant isomorphism

$$R\Gamma_A(\mathfrak{X}) \otimes^L_{A_{inf}} A_{crys} \simeq R\Gamma_{crys}(\mathfrak{X}_{\mathcal{O}_C/p}), \tag{4.9}$$

[6]More precisely, given a finitely presented torsion $\mathcal{O}_C$-module M, there is a unique way to define a number $\ell_{\mathcal{O}_C}(M) \in \mathbf{R}_{\geq 0}$ that behaves additively under short exact sequences, and carries $\mathcal{O}_C/p$ to 1. A high-brow perspective on this length arises from algebraic K-theory: by the excision sequence for $\mathcal{O}_C \to C$, one may identify K_0 of the category of finitely presented torsion $\mathcal{O}_C$-modules with $K_1(C)/K_1(\mathcal{O}_C) \simeq C^*/\mathcal{O}_C^*$. Postcomposing with the p-adic valuation map $C^*/\mathcal{O}_C^* \to \mathbf{R}$ (normalized to send p to 1) gives the desired normalized length function; see also [Ce, §4.10].

which is the absolute crystalline comparison theorem. This isomorphism can be then used to prove the crystalline comparison theorem, see [BMS2, Theorem 14.5].

6. *Bounding the failure of integral comparison maps to be isomorphisms.* Consider the element

$$\xi = \mu/\phi^{-1}(\mu) = \frac{[\epsilon]-1}{[\epsilon^{\frac{1}{p}}]-1} = \sum_{i=0}^{p-1}[\epsilon^{\frac{i}{p}}].$$

This element can be checked to be a generator for $\ker(\theta)$, and thus $\phi(\xi)$ generates $\ker(\widetilde{\theta})$. We also have the formula $\mu = \xi \cdot \phi^{-1}(\mu)$ which provides justification for the heuristic formula "$\mu = \prod_{n\geq 0}\phi^{-n}(\xi)$." The zero locus of μ is depicted in orange in Figure 4.1.

The construction of $R\Gamma_A(\mathfrak{X})$ shows that there is a naturally defined map

$$R\Gamma_A(\mathfrak{X}) \to R\Gamma(X_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{inf}$$

in the almost category that has an inverse up to μ^d, where $d = \dim(X)$. Specializing along the natural map $A_{inf} \to A_{crys}$ and using (5), we obtain a naturally defined almost map

$$R\Gamma_{crys}(\mathfrak{X}_{\mathcal{O}_C/p}) \to R\Gamma(X_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{crys}$$

which is also invertible up to μ^d. In particular, one has reasonable control on the failure of the integral comparison maps to be isomorphism, as in the work of Faltings [Fa3, Fa4].

7. *Recovering crystalline cohomology of the special fibre from the generic fibre, integrally.* Each cohomology group M of $R\Gamma_A(\mathfrak{X})$ can be shown to be a finitely presented A_{inf}-module equipped with a map $\phi_M : \phi^* M \to M$ that is an isomorphism outside $\mathrm{Spec}(\mathcal{O}_C) \overset{\widetilde{\theta}}{\to} \mathrm{Spec}(A_{inf})$, and is free after inverting p; such pairs (M, ϕ_M) are analogues over C of the Breuil-Kisin modules from [Ki], were introduced and studied by Fargues, and were called Breuil-Kisin-Fargues modules in [BMS2, §4.3]. Using some abstract properties of such modules and Theorem 4.2.1, one can show the following: if $H^i_{crys}(\mathfrak{X}_k/W(k))$ and $H^{i+1}_{crys}(\mathfrak{X}_k/W(k))$ are torsionfree, then $H^i_{crys}(\mathfrak{X}_k/W(k))$ is determined functorially from the generic fibre X (see [BMS2, Theorem 1.4]). In particular, in naturally arising geometric situations (such as K3 surfaces), this implies that for different good models for the same generic fibre X, the integral crystalline cohomology of the special fibres is independent of the choice of good model.

4.3 Strategy of the proof

Theorem 4.2.1 posits the existence of an A_{inf}-valued cohomology theory attached to $\mathfrak{X}$. A natural way to construct such a theory is to work *locally* on $\mathfrak{X}$, i.e., construct a complex $A\Omega_{\mathfrak{X}}$ of sheaves of A_{inf}-modules on the formal scheme $\mathfrak{X}$, and try to prove all the comparisons in Theorem 4.2.1 at the level of sheaves. With one caveat, this is essentially how the construction goes.

The necessary tools are:

1. *The nearby cycles map.* Breaking from the notation used in §3, we write $\nu : X_{proet} \to \mathfrak{X}$ for the *nearby cycles map*; this it the map on topoi whose pullback is induced by the observation that if $\mathfrak{U} \subset \mathfrak{X}$ is an open subset, then we get a rational open subset $U \subset X$ on passage to generic fibres. The reason behind calling this map the "nearby cycles map" name is a theorem of Huber [Hu2, Theorem 0.7.7]: for any integer n, the stalk of $R\nu_* \mathbf{Z}/n$ at a point $x \in \mathfrak{X}$ is given by the cohomology of the "nearby fiber", or the "Milnor fiber", i.e., by $R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{\mathfrak{X},x}[\frac{1}{p}])_{et}, \mathbf{Z}/n)$.

2. *The pro-étale sheaf $A_{inf,X}$.* Fontaine's construction of $A_{inf}(R) := W(R^\flat)$ and the map $\theta : A_{inf}(R) \to R$ makes sense for any ring p-adically complete ring R (see Remark 3.1.8). In particular, this yields a presheaf $A_{inf,X} := A_{inf}(\mathcal{O}_X^+)$ of A_{inf}-modules on the pro-étale site of X. Using the locally perfectoid nature of X_{proet} from Theorem 3.2.2, this presheaf can be checked to be a sheaf. By a variant of the primitive comparison theorem (see Theorem 3.0.2), the cohomology of $A_{inf,X}$ is almost isomorphic to $H^*(X_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{inf}$; as we shall see, this is the only place where properness enters the proof of Theorem 4.2.1.

3. *Killing torsion in the derived category.* Given a ring A and a nonzerodivisor $f \in A$, we need a systematic technique for killing the f-torsion in the homology of a chain complex K of A-modules; the adjective "systematic" means roughly that the construction should only depend on the class of K in the derived category $D(A)$. While this is impossible to achieve with an *exact* functor $D(A) \to D(A)$, the following non-exact functor on chain complexes does the job: given a chain complex $K^\bullet$ of f-torsionfree A-modules, define a new chain complex $\eta_f K^\bullet$ as a subcomplex of $K^\bullet[\frac{1}{f}]$ with the following terms:
$$(\eta_f K^\bullet)^i = \{\alpha \in f^i K^i \mid d(\alpha) \in f^{i+1} K^{i+1}\}.$$
One easily checks that $H^i(\eta_f K^\bullet)$ identifies with $H^i(K^\bullet)/(f\text{-torsion})$, and thus the association $K^\bullet \to \eta_f K^\bullet$ derives to give a functor $L\eta_f : D(A) \to D(A)$. This construction is motivated by ideas of Berthelot-Ogus in crystalline cohomology [BO2, §8], can be thought of as a "decalage" of the f-adic filtraton on K in the sense of Deligne [De1], and discussed in much more depth in [BMS2, §6], [Bh, §6], [Mor, §2].

With these tools in play, here are the two main steps in the construction:

1. *The first approximation.* Consider the complex $A\Omega^{pre}_{\mathfrak{X}} := R\nu_* A_{inf,X}$ as an object of the derived category $D(\mathfrak{X}, A_{inf})$ of A_{inf}-modules on the formal scheme $\mathfrak{X}$. As explained above, we have
$$R\Gamma(\mathfrak{X}, R\nu_* A_{inf,X}) = R\Gamma(X_{proet}, A_{inf,X}) \stackrel{a}{\simeq} R\Gamma(X_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{inf}.$$
As almost zero modules die[7] after base change along $A_{inf} \to W(C^\flat)$, this tells us that the complex $R\Gamma(\mathfrak{X}, A\Omega^{pre}_{\mathfrak{X}})$ satisfies Theorem 4.2.1 (1). Now let's instead consider the Hodge-Tate specialization $\widetilde{\theta}^* R\Gamma(\mathfrak{X}, A\Omega^{pre}_{\mathfrak{X}})$, where $\widetilde{\theta} = \theta \circ \phi^{-1} : A_{inf} \to \mathcal{O}_C$. By formal nonsense with the projection formula, this complex identifies with the $\mathcal{O}_C$-complex $R\Gamma(X_{proet}, \widehat{\mathcal{O}_X^+})$, viewed as an A_{inf}-complex via $\widetilde{\theta}$. To

[7]In terms of Figure 4.1, the locus where almost zero modules lives is the crystalline specialization, which does not intersect the locus defined by $\mathrm{Spec}(W(C^\flat)) \to \mathrm{Spec}(A_{inf})$ the étale specialization.

compute this explicitly, assume further that $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_C\langle t^{\pm 1}\rangle)$ is the formal torus. One can then essentially repeat the calculation given in Lemma 3.3.1 to obtain that

$$\begin{aligned} R\Gamma(X_{proet}, \widehat{\mathcal{O}_X^+}) &\simeq \widehat{\bigoplus_{i\in\mathbf{Z}[\frac{1}{p}]}}\Big(\mathcal{O}_C\cdot T^i \xrightarrow{T^i\mapsto(\underline{\epsilon}^i-1)T^i} \mathcal{O}_C\cdot T^i\Big) \\ &\simeq \widehat{\bigoplus_{i\in\mathbf{Z}}}\Big(\mathcal{O}_C\cdot T^i \xrightarrow{T^i\mapsto(\underline{\epsilon}^i-1)T^i} \mathcal{O}_C\cdot T^i\Big) \oplus \widehat{\bigoplus_{i\in\mathbf{Z}[\frac{1}{p}]-\mathbf{Z}}}\Big(\mathcal{O}_C\cdot T^i \xrightarrow{T^i\mapsto(\underline{\epsilon}^i-1)T^i} \mathcal{O}_C\cdot T^i\Big) \\ &\simeq \widehat{\bigoplus_{i\in\mathbf{Z}}}\Big(\mathcal{O}_C\cdot T^i \xrightarrow{0} \mathcal{O}_C\cdot T^i\Big) \oplus \mathrm{Err}, \end{aligned} \tag{4.10}$$

where Err is an $\mathcal{O}_C$-complex whose homology is killed by $\epsilon^{\frac{1}{p}} - 1$ (since $\epsilon^i - 1 \mid \epsilon^{\frac{1}{p}} - 1$ for any $i \in \mathbf{Z}[\frac{1}{p}] - \mathbf{Z}$). Thus, when viewed as an A_{inf}-complex via $\widetilde{\theta}$, this tells us that $R\Gamma(X_{proet}, \widehat{\mathcal{O}_X^+})$ looks like it has the right size for the Hodge-Tate comparison, up to an error term Err whose homology is killed by $\mu := [\underline{\epsilon}] - 1$. One can also repeat the same calculation without specializing to compute $R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}^{pre})$ directly in this case[8] to see that the error term Err above comes from an analogous summand of $R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}^{pre})$ whose homology is also μ-torsion. Thus, we want to modify $A\Omega_{\mathfrak{X}}^{pre}$ in a manner that functorially kill the μ-torsion in its homology.

2. *The main construction.* The preceding analysis suggests defining

$$A\Omega_{\mathfrak{X}} := L\eta_\mu A\Omega_{\mathfrak{X}}^{pret} := L\eta_\mu R\nu_* A_{inf,X} \quad \text{and} \quad R\Gamma_A(\mathfrak{X}) := R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}})$$

In this definition, the Frobenius $\phi_{\mathfrak{X}}$ is induced by the sequence

$$\phi^*(A\Omega_{\mathfrak{X}}) \simeq L\eta_{\phi(\mu)}\phi^* R\nu_* A_{inf,X} \simeq L\eta_{\phi(\xi)} L\eta_\mu R\nu_* A_{inf,X} \to L\eta_\mu R\nu_* A_{inf,X} =: A\Omega_{\mathfrak{X}},$$

where the first isomorphism is by "transport of structure", the second isomorphism relies on a transitivity property of the $L\eta$-functor (namely, $L\eta_f \circ L\eta_g \simeq L\eta_{fg}$ with obvious notation), the third map exists because of the structure of $R\nu_* A_{inf,X}$ (namely, the construction of $L\eta_f$ shows that if K can be represented by a chain complex $K^\bullet$ of f-torsionfree modules with $K^i = 0$ for $i < 0$, then there is an evident map $L\eta_f(K) \to K$) and the fact that $\phi^* A_{inf,X} \simeq A_{inf,X}$, and the last isomorphism is a definition.

This definition does indeed work, and we only briefly indicate what goes into proving the required comparison isomorphisms:

- Étale cohomology. We have already explained in (1) above why $R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}^{pre})$ satisfies the requisite comparison isomorphism with étale cohomology after base change to $W(C^\flat)$. The rest follows immediately $L\eta_\mu(K)$ and K are naturally isomorphic after inverting μ for any complex K.

[8] The entire calculation remains the same: one simply replaces $\mathcal{O}_C$ with A_{inf} in the formulas above, and one is not allowed to simplify the differential on the first summand to 0 as $[\underline{\epsilon}]^i - 1$ is not zero on A_{inf} for $i \in \mathbf{Z}$.

- Hodge-Tate cohomology. This comparison was essentially forced to be true by the calculation in (1) above. More precisely, one defines a map a $\Omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\} \to \mathcal{H}^1(\widetilde{\theta}^* A\Omega_{\mathfrak{X}})$ via a variant of the construction in §3.4, and then checks that it yields isomorphisms

$$\Omega^i_{\mathfrak{X}/\mathcal{O}_C}\{-i\} \simeq \mathcal{H}^i(\widetilde{\theta}^* A\Omega_{\mathfrak{X}})$$

by unraveling the preceding map and matching it with the computation in (4). The Hodge-Tate spectral sequence is then simply the standard spectral sequence expressing the hypercohomology of a complex of sheaves in terms of the hypercohomology of its cohomology sheaves. We refer to [BMS2, §8], [Bh, §6] for more details.

- de Rham cohomology. This comparison results from the previous one using the following observation:

Proposition 4.3.1. *For any ring A with a nonzerodivisor $f \in A$ and a complex $K \in D(A)$, the complex $L\eta_f K/f$ is naturally represented by the chain complex*

$$(H^*(K/f), \mathrm{Bock}_f) := \Big(.... \to H^i(f^i K/f^{i+1}K) \xrightarrow{\mathrm{Bock}_f} H^{i+1}(f^{i+1}K/f^{i+2}K) \to ....\Big),$$

where Bock_f is the boundary map "Bockstein" on cohomology associated to the exact triangle

$$f^{i+1}K/f^{i+2}K \xrightarrow{\mu} f^i K/f^{i+2}K \xrightarrow{std} f^i K/f^{i+1}K$$

in $D(A)$. Moreover, when K admits the structure of a commutative algebra in $D(A)$, the preceding identification naturally makes $L\eta_f(K)/f$ into a differential graded algebra via cup products.

We apply this observation to $K = L\eta_{\phi^{-1}\mu} R\nu_* A_{inf,X}$ and $f = \xi = \mu/\phi^{-1}(\mu)$ is the displayed generator of $\ker(\theta)$. Note that $A\Omega_{\mathfrak{X}} \simeq L\eta_\xi(K)$. Applying the previous observation tells us that $\theta^* A\Omega_{\mathfrak{X}} \simeq A\Omega_{\mathfrak{X}}/\xi$ is naturally represented by the differential graded algebra $(H^*(K/\xi), \mathrm{Bock}_\xi)$. The complex K is a Frobenius twist of $A\Omega_{\mathfrak{X}}$; keeping track of the twists, one learns that K/ξ is the Hodge-Tate specialization $\widetilde{\theta}^* A\Omega_{\mathfrak{X}}$. Thus, by the previous comparison, the i-th term of $H^*(K/\xi)$ is thus given by

$$\Omega^i_{\mathfrak{X}/\mathcal{O}_C}\{-i\} \otimes_{\mathcal{O}_C} \xi^i/\xi^{i+1} \simeq \Omega^i_{\mathfrak{X}/\mathcal{O}_C},$$

i.e., by differential forms. Unraveling these isomorphisms, the Bockstein differential Bock_ξ can then be checked to coincide with the de Rham differential, thus proving that $A\Omega_{\mathfrak{X}}/\xi \simeq \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C}$. We refer to [Mor, Theorem 5.9] and [Bh, Proposition 7.9] for more details on the implementation of this approach.

- Crystalline cohomology. There are two possible approaches here: one either repeats the arguments given for the de Rham comparison above using de Rham-Witt complexes to identify $A\Omega_{\mathfrak{X}}/\mu$ with the relative de Rham-Witt complex of $\mathfrak{X}/\mathcal{O}$, or one directly proves that $A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{inf}} A_{crys}$ identifies with the absolute crystalline cohomology of $\mathfrak{X}$ over A_{crys}. Both approaches yield strictly finer statements than Theorem 4.2.1 (4). We refer to [BMS2, §11], [Mor] for the first approach, and [BMS2, §12] for the second approach.

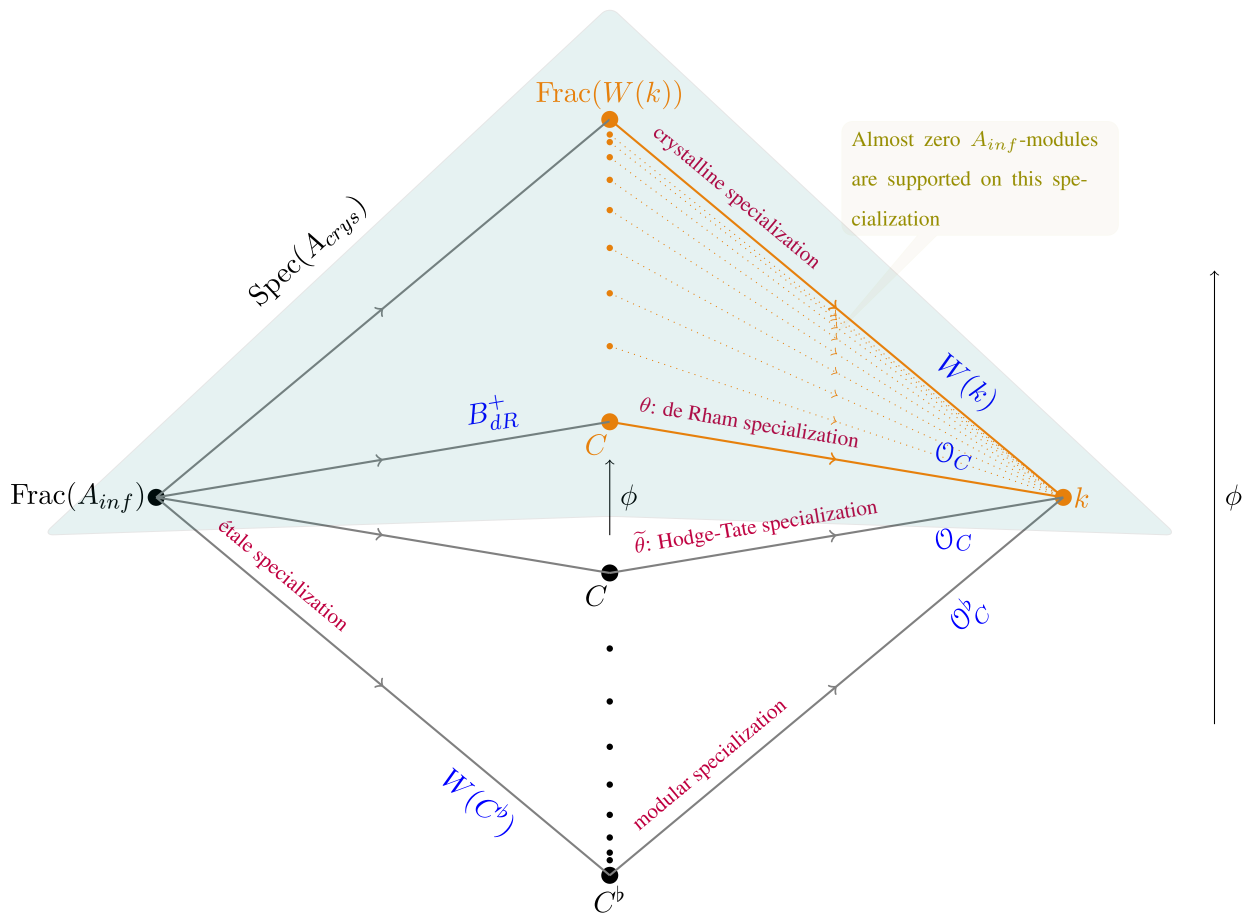


Figure 4.1: A cartoon of $\mathrm{Spec}(A_{inf})$. This depiction of the poset of prime ideals in A_{inf} emphasizes certain vertices and edges that are relevant to p-adic cohomology theories.

- The darkened vertices (labelled '●' or '●') indicate (certain) points of $\mathrm{Spec}(A_{inf})$ and are labelled by the corresponding residue field.
- The gray/orange arrows indicate specializations in the spectrum, while the blue label indicates the completed local ring along the specialization.
- The arrow labelled ϕ on the far right indicates the Frobenius action on $\mathrm{Spec}(A_{inf})$, which fixes the 4 vertices of the outer diamond in the above picture.
- The labels in purple match the arrows to one of the specializations that are important for p-adic comparison theorems.
- The smaller bullets (labelled '•' or '•') down the middle are meant to denote the $\phi^{\mathbf{Z}}$-translates to the two drawn points labelled C (with $\phi^{\mathbf{Z}_{\geq 0}}$ translates of the generic point of the de Rham specialization in orange, and the rest in black), and are there to remind the reader that not all points/specialization in $\mathrm{Spec}(A_{inf})$ have been drawn.
- The vertices/labels/arrows in orange mark the points and specializations that lie in $\mathrm{Spec}(A_{inf}/\mu) \subset \mathrm{Spec}(A_{inf})$.

- The triangular region covered in teal identifies the image of $\mathrm{Spec}(A_{crys}) \to \mathrm{Spec}(A_{inf})$.

5. Exercises

This section was written jointly with Daniel Litt.

Using the Hodge-Tate decomposition

1. Calculate $h^{i,j}(X)$ (in the sense of Deligne's mixed Hodge theory) for the following varieties X by using the Hodge-Tate decomposition and calculating the corresponding étale cohomology groups (as Galois modules) first.

   (a) $X = \mathbf{Gr}(k, n)$ is a Grassmannian.

   (b) X is a smooth affine curve.

   (c) $X = \mathbf{P}^1/\{0, \infty\}$ is nodal rational curve.

   (d) $X \subset \mathbf{P}^2$ is a cubic curve with 1 cusp.

2. Let R be a finitely-generated integral $\mathbf{Z}$-algebra with fraction field K, and let X, Y be smooth proper R-schemes. Suppose that if $\mathfrak{p}$ is any closed point of $\operatorname{Spec}(R)$, and $k/\kappa(\mathfrak{p})$ is any finite extension, then $\#|X(k)| = \#|Y(k)|$.

   (a) Use the Hodge-Tate decomposition to show that $h^{i,j}(X_K) = h^{i,j}(Y_K)$ for all X, Y. (Hint: Use the Lefschetz fixed-point formula to figure out how Frobenii act; use Chebotarev to conclude that the Galois representations on the cohomology of X and Y are the same. Use the Hodge-Tate decomposition to finish the proof.)

   (b) * Let X, Y be birational Calabi-Yau varieties over the complex numbers (i.e. varieties with trivial canonical bundle). Show that they have the same Hodge numbers. (Hint: Use p-adic integration to count points of reductions.)

3. The goal of this exercise is to use the Hodge-Tate decomposition to translate a point-counting statement to a geometric one[1]. Let $X/\mathbf{C}$ be a smooth projective variety that is defined over $\mathbf{Q}$. For a prime p, write X_p for a reduction of X to $\overline{\mathbf{F}_p}$; this makes sense for all but finitely many p's once an integral model of X has been chosen. Assume that there exists a polynomial P_X such that for all but finitely many p, we have $P_X(p) = \#X(\mathbf{F}_p)$. We shall

[1] We restrict ourselves to working over $\mathbf{Q}$ to avoid notational complications. The general version of the result in this exercise is due Katz, see [HR, Appendix].

(a) Show that for each n, the $G_{\mathbf{Q}}$-representation $H^n(X, \mathbf{Q}_\ell)$ is isomorphic to a direct sum of copies of $\mathbf{Q}_\ell(-i)$ up to semisimplification. (Hint: use the Weil conjectures and Chebotarev.)

(b) Show that $h^{i,j}(X) = 0$ for $i \neq j$. (Hint: use the Hodge-Tate decomposition.)

We also encourage the reader to think about the converse assertion: if $h^{i,j}(X) = 0$ for $i \neq j$, then is the function $p \mapsto \#X(\mathbf{F}_p)$ given by a polynomial, at least on a large set of primes? (Hint: try to use the "Newton-lies-above-Hodge" theorem.)

Inverse limits of schemes and perfectoid abelian varieties

4. Let $\{X_i\}$ be a cofiltered system of quasi-compact and quasi-separated schemes with affine transition maps $f_{ij} : X_i \to X_j$.

   (a) Show that the inverse limit $X_\infty := \lim_i X_i$ exists in the category of schemes, and coincides with the inverse limit in the category of locally ringed spaces. Write $f_i : X_\infty \to X_i$ for the projection map.

   (b) For any quasi-coherent sheaf $\mathcal{F}$ on some X_0, show that the natural pullback induces a isomorphisms
   $$H^*(X_\infty, f_i^*\mathcal{F}) \simeq \operatorname*{colim}_{f_{i0}:X_i \to X_0} H^*(X_i, f_{i0}^*\mathcal{F}).$$

   Much more material on such limits can be gleaned from [SP, Tag 01YT].

5. Let k be an algebraically closed field and A/k an abeliann variety of dimension g. The purpose of this problem is to show that A is a $K(\pi, 1)$.

   (a) Show that any connected finite étale cover of A is also an abeliann variety (note that this is not true for commutative group schemes which are not proper – find a commutative group scheme with a connected finite étale cover which does not admit the structure of a group scheme).

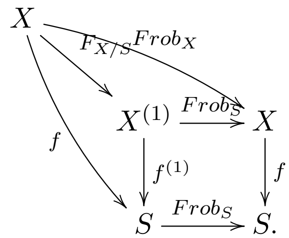
   (b) Deduce from the previous part that the étale fundamental group of A is canonically isomorphic to its Tate module.

   (c) Let B be an abelian group. Show that any class in $H^1(A_{\text{ét}}, B)$ is killed by some finite étale cover of A.

   (d) Observe that if $R = \mathbf{F}_q$ is a finite field, the ring $H^*(A_{\text{ét}}, R)$ is given a Hopf-algebra structure by the multiplication on A. Conclude that if the characteristic of R is different from that of k, then $H^*(A_{\text{ét}}, R)$ is an exterior algebra on $2g$ generators in degree 1. What happens if the characteristic of R equals that of k?

   (e) Deduce from the previous part the following fact: for any finite abeliann group B, the natural map
   $$H^*(\pi_1^{\text{ét}}(A), B) \to H^*(A_{\text{ét}}, B)$$
   is an isomorphism.

6. The goal of this exercise is to sketch why the inverse limit of multiplication by p on an abelian scheme over $\mathcal{O}_C$ gives a perfectoid space. For this exercise, we shall need the relative Frobenius map: if S is a scheme of characteristic p, and $f : X \to S$ is a map, then we define the Frobenius twist $X^{(1)} := X \times_{Frob_S, S} S$ as the base change of f along the Frobenius on S, and write $F_{X/S} : X \to X^{(1)}$ for map induced by the Frobenius on X. This fits into the following diagram:



Given a flat[2] map $f : X \to S$, we shall say that f is *relatively perfect* if $F_{X/S}$ is an isomorphism. Note that the functor $X \mapsto X^{(1)}$ on S-schemes preserves finite limits, and thus carries (commutative) group schemes to (commutative) group schemes.

(a) Let R be a p-adically complete and p-torsionfree $\mathcal{O}_C$-algebra such that the map $\mathrm{Spec}(R/p) \to \mathrm{Spec}(\mathcal{O}_C/p)$ relatively perfect. Show that $R[\frac{1}{p}]$ is naturally a perfectoid algebra.

(b) Let A be a ring of characteristic p, and let G be a finite flat group scheme over A. Assume that the relative Frobenius map $G \to G^{(1)}$ is the trivial map. Using Verschiebung, show that G is killed by p. Deduce the following: if H is a smooth group scheme over A, then the relative Frobenius map $H \to H^{(1)}$ factors multiplication by p on H.

(c) Let A be a ring of characteristic p. Let $\mathcal{A}$ be an abelian scheme over A. Show that the inverse limit of multiplication by p on $\mathcal{A}$ is relatively perfect over A.

(d) Let $\mathcal{A}/\mathcal{O}_C$ be a smooth abelian group scheme with generic fiber A. Show that the inverse limit $\lim_p A$ of multiplication by p on A is naturally a perfectoid space.

(e) Let $\mathcal{A}/\mathcal{O}_C$ be a smooth abelian group scheme. Show that the p-adic completion of the inverse limit $\lim_p \mathcal{A}$ depends only on the abelian $\mathcal{O}_C/p$-scheme $\mathcal{A} \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$.

Derived completions of complexes

7. For any complex K of torsionfree abelian groups, define $\widehat{K} := \lim K/p^n K$.

(a) Show that the operation $K \mapsto \widehat{K}$ passes to the derived category $D(\mathrm{Ab})$ of abelian groups, i.e., it carries quasi-isomorphisms of chain complexes to quasi-isomorphisms. We write the resulting functor $D(\mathrm{Ab}) \to D(\mathrm{Ab})$ also by $K \mapsto \widehat{K}$, and call it the p-adic completion functor.

(b) Show that the p-adic completion functor is given by the formula

$$K \mapsto R\lim_n (K \otimes^L_{\mathbf{Z}} \mathbf{Z}/p^n).$$

[2]More generally, it is convenient to adopt the same terminology if f and $Frob_S$ are Tor-independent.

(c) Show that the p-adic completion functor is exact, i.e., preserves exact triangles.

(d) Show that $\widehat{\widehat{K}} \simeq \widehat{K}$, i.e., the completion is complete.

(e) Show that $K \in D(\mathrm{Ab})$ is complete (i.e., $K \simeq \widehat{K}$) if and only if $\mathrm{RHom}(\mathbf{Z}[\frac{1}{p}], K) \simeq 0$.

(f) Prove Nakayama's lemma: for $K \in D(\mathrm{Ab})$, if $K \otimes^L_{\mathbf{Z}} \mathbf{Z}/p \simeq 0$, then $\widehat{K} \simeq 0$.

(g) If A is a p-divisible abelian group, show that $\widehat{A} \simeq T_p(A)[1]$, where $T_p(A)$ is the Tate module.

The cotangent complex, perfect rings, perfectoid rings

8. Let $A \to B$ be an lci map of rings, i.e., after Zariski localization on both rings, the map factors as $A \xrightarrow{a} P \xrightarrow{b} B$, where a is a polynomial extension, and b is a quotient defined by a regular sequence.

   (a) Show that $H^1(L_{B/A})$ is torsionfree (i.e., not killed by a nonzerodivisor on B).

   (b) Show that if $A \to B$ is flat and $f \in A$ is a nonzerodivisor with $A[\frac{1}{f}] \to B[\frac{1}{f}]$ smooth, then $L_{B/A} \simeq \Omega^1_{B/A}$.

   (c) Let $K/\mathbf{Q}_p$ be a nonarchimedean extension, and let L/K be an algebraic extension. Show that $L_{\mathcal{O}_L/\mathcal{O}_K} \simeq \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$.

9. Let A be a perfect $\mathbf{F}_p$-algebra.

   (a) Use the "transitivity triangle" to show that $L_{A/\mathbf{F}_p} = 0$.

   (b) Deduce that A admits a unique flat deformation over $\mathbf{Z}/p^n$ for any n.

   (c) Using (a), show that the derived p-adic completion of $L_{W(A)/\mathbf{Z}_p}$ vanishes. Convince yourself that it is necessary to take a completion here.

   (d) Using the transitivity triangle, show that for any map $A \to B$ of perfect $\mathbf{F}_p$-algebras, the derived p-adic completion of $L_{W(B)/W(A)}$ vanishes.

   (e) More generally, if $R \to S$ is a map of p-torsionfree $\mathbf{Z}_p$-algebras such that $R/p \to S/p$ is relatively perfect, show that the p-adic completion of $L_{S/R}$ vanishes.

10. Let $A \to B$ be a map of integral perfectoid rings.

   (a) Show that the square

$$\begin{array}{ccc} W(A) & \longrightarrow & W(B) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} \\ A & \longrightarrow & B \end{array}$$

   is a pushout square of commutative rings. (Hint: use [BMS2, Remark 3.11]).

   (b) Show that the derived p-adic completion of $L_{B/A}$ vanishes.

11. Give examples of:

   (a) Give an example of a map $A \to B$ of finite type $\mathbf{C}$-algebras where $L_{B/A} \in D^{\leq -2}(B)$, i.e., $H^i(L_{B/A}) = 0$ for $i \geq -1$.

(b) A p-adically complete ring A such that A/p is semiperfect, but $W(A) \xrightarrow{\theta} A$ does not have a principal kernel.

(c) A semiperfect $\mathbf{F}_p$-algebra A such that $L_{A/\mathbf{F}_p}$ is nonzero.

(d) (*) An $\mathbf{F}_p$-algebra A such that $L_{A/\mathbf{F}_p} = 0$, but A is not perfect.

12. This exercise is meant to illustrate a general feature of certain valuation rings, and is not relevant to the rest of these notes. Let $\mathbf{Z}_p \to V$ be a faithfully flat map with V a valuation ring. Assume that $\mathrm{Frac}(V)$ is algebraically closed.

(a) (*) Show that V can be written as a filtered colimit of regular $\mathbf{Z}_p$-algebras. (Hint: use de Jong's alterations theorem from [dJ]).

(b) Deduce that $V[\frac{1}{p}]$ is ind-smooth over $\mathbf{Q}_p$. (This can be proven without using (a)).

(c) Show that any regular $\mathbf{Z}_p$-algebra is lci over $\mathbf{Z}_p$.

(d) Deduce that $L_{V/\mathbf{Z}_p} \simeq \Omega^1_{V/\mathbf{Z}_p}$.

Group cohomology and the pro-étale site

13. Fix a finite group G. Let X be a topological space equipped with an action of G, and let $f : X \to Y$ be a G-equivariant map (for the trivial G-action on Y). Let A be a coefficient ring.

(a) Show that the natural pullback $H^0(Y, A) \to H^0(X, A)$ has image contained inside the G-invariants $H^0(X, A)^G$. Using the spectral sequence for a composition of derived functors, deduce that there is a natural map $H^i(Y, A)$ to groups $H^i_G(X, A)$ which are computed by a Hochschild-Serre spectral sequence

$$E_2^{i,j} : H^i(G, H^j(X, A)) \Rightarrow H^{i+j}_G(X, A).$$

(b) Lift the preceding assertion to construct a natural map

$$R\Gamma(Y, A) \to R\Gamma(G, R\Gamma(X, A))$$

in the derived category $D(A)$.

(c) If f is a G-torsor (i.e., f realizes Y as the quotient of X by G, and the G-action has no non-trivial stabilizers on X), then show that the maps above are isomorphisms, i.e., we have

$$H^i(Y, A) \simeq H^i_G(X, A) \quad \text{and} \quad R\Gamma(Y, A) \simeq R\Gamma(G, R\Gamma(X, A)).$$

(d) Assume that X is contractible, and that f is a G-torsor. Show that the above maps identify $H^*(X, A)$ with the group cohomology $H^*(G, A)$ of G.

14. The goal of this exercise is to show that the ideas going into the construction of the pro-étale site lead to a sheaf-theoretic perspective on continuous cohomology, at least with a large class of coefficients; see [BS, §4.3], [Sc2, §3, erratum] for more. Let G be a profinite group. Let $\mathcal{C}_G$ be the category of sets equipped with a continuous G-action. Equip $\mathcal{C}_G$ with the structure of a site by declaring all continuous surjective maps to be covers. Write $H^*(\mathcal{C}_G, -)$ for the derived functors of $\mathcal{F} \mapsto \mathcal{F}(*)$, where $*$ is the 1 point set with the trivial G-action.

(a) Let X be a topological space equipped with a continuous G-action. Show that $\mathrm{Hom}_G(-, X)$ defines a sheaf on $\mathcal{C}_G$. We write $\mathcal{F}_X$ for this sheaf; if X is a G-module, then $\mathcal{F}_X$ is naturally a sheaf of abelian groups, likewise for rings, etc..

(b) Let A be a topological abelian group equipped with a continuous G-action. By considering the Cech nerve of the continuous G-equivariant map $G \to *$, show that there is a canonical map

$$c_A : H^*_{cts}(G, A) \to H^*(\mathcal{C}_G, \mathcal{F}_A).$$

Write $\mathcal{D}$ for the category of all A such that c_A is an isomorphism.

(c) Show that any discrete G-module lies in $\mathcal{D}$. (Hint: first show the analogous assertion for the category $\mathcal{C}^f_G$ of finite G-sets with a continuous G-action, and then analyze the natural morphism $\mathrm{Sh}(\mathcal{C}_G) \to \mathrm{Sh}(\mathcal{C}^f_G)$ on the categories of sheaves.)

(d) Fix a sequence

$$M_1 \to M_2 \to ... \to M_n \stackrel{f_n}{\to} M_{n+1}....$$

in $\mathcal{D}$ with M_i being Hausdorff and the f_n's being closed immersions. Show that the colimit $\mathrm{colim}_i\, M_i$ also belongs to $\mathcal{D}$.

(e) Fix a sequence

$$...M_{n+1} \stackrel{f_n}{\to} M_n \to .... \to M_2 \to M_1 \to M_0 = 0$$

Assume that each f_n has sections after base change along a continuous map $K \to M_i$ with K a profinite set, and that $\ker(f_n) \in \mathcal{D}$ for all $n \geq 1$. Then $\lim_n M_n \in \mathcal{D}$.

(f) Fix a finite extension $K/\mathbf{Q}_p$. Let $G = \mathrm{Gal}(\overline{K}/K)$. Fix a completed algebraic closure $\mathbf{C}_p$ of K, and let V be a finite dimensional $\mathbf{C}_p$-vector space with a continuous semilinear G-action. Show that $V \in \mathcal{D}$.

15. Let $G = \oplus_{i=1}^n \mathbf{Z}_p \cdot \gamma_i$ be a finite free $\mathbf{Z}_p$-module with generators γ_i; we view G as a profinite group. Let M be a discrete G-module. Show that $H^*_{cts}(G, M)$ is computed as the cohomology of the complex

$$\otimes_{i=1}^n \Big(M \stackrel{\gamma_i - 1}{\to} M\Big).$$

Étale and de Rham cohomology in equicharacteristic p

16. Let k be a field of characteristic $p > 0$ and X a k-variety. Compute $H^1(X_{\text{ét}}, \mathbf{F}_p)$ if

(a) $X = \mathbf{A}^1_k$ (Hint: Use the Artin-Schreier exact sequence).

(b) X is a smooth, proper, geometrically connected curve of genus 1 (Hint: The answer depends on the curve).

17. Let X be a smooth variety over a perfect field k of characteristic $p > 0$.

(a) Suppose X admits a flat lift X' to $W_2(k)$, and that Frobenius lifts to X'. Show that the Cartier isomorphism lifts to a map of complexes

$$\Omega^1_{X^{(p)}/k} \to F_* \Omega^\bullet_{X/k}.$$

(b) In the situation above, let F_1, F_2 be two different lifts of Frobenius. Show that the maps constructed in (a) using these two lifts are homotopic.

(c) Now suppose that X lifts to $W_2(k)$, but do not assume that Frobenius lifts. Show that the Cartier isomorphism lifts to a map

$$\Omega^1_{X^{(p)}/k} \to F_* \Omega^\bullet_{X/k}$$

in $D^b(X)$. (Hint: Cover X by affines and use a Cech complex.)

p-adic Hodge theory

18. Let X be a commutative group scheme over $\mathscr{O}_{\mathbf{C}_p}$.

(a) Use the construction of the Hodge-Tate comparison map to define a pairing

$$\int : T_p(X) \times H^1_{dR}(X) \to \mathbf{C}_p(1).$$

(b) One can think of the above pairing as "integrating a form along a (closed) cycle." What is the analogue of a path integral?

(c) In the case $X = \mathbf{G}_m$, make everything as explicit as you can.

19. Let C be a complete and algebraically closed extension of $\mathbf{Q}_p$. Let $K/\mathbf{Q}_p$ be a finite extension that is contained in C. Recall that there is a natural surjective map $A_{inf} \xrightarrow{\theta} \mathcal{O}_C$. Write $B^+_{dR} \to C$ for map obtained from the previous one by inverting p and completing, i.e., B^+_{dR} is the completion of $A_{inf}[\frac{1}{p}]$ along $\ker(\theta[\frac{1}{p}])$.

(a) Show that the map $\mathcal{O}_K \to \mathcal{O}_C$ lifts across $A_{inf} \to \mathcal{O}_C$ if and only if $K/\mathbf{Q}_p$ is unramified.

(b) Show that the map $K \to C$ always lifts uniquely across $B^+_{dR} \to C$.

Now let X_0/K be a smooth rigid space, and let X/C denote its base change.

(c) Using the deformation theoretic interpretation from the notes, show that the complex $\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}_X}$ on X_{proet} splits for X as above.

6. Projects

This section was written jointly with Matthew Morrow. Let C be a complete and algebraically closed extension of $\mathbf{Q}_p$.

1. **Understand the Hodge-Tate filtration for singularities**[1]**.** The primitive comparison theorem holds true for non-smooth spaces X as well. Thus, for X proper, we still have a "Hodge-Tate" spectral sequence
$$E_2^{i,j} : H^i(X, R^j\nu_*\widehat{\mathcal{O}_X}) \Rightarrow H^{i+j}(X, C).$$
It is thus of interest to understand the sheaves $R^j\nu_*\widehat{\mathcal{O}_X}$. This problem turns out to be closely related to the singularities of X. Recall first that a ring R is called *semi-normal* if and only if, for any $y, z \in R$ satisfying $y^3 = z^2$, there exists a unique $x \in R$ satisfying $x^2 = y$, $x^3 = z$. A relevant source for the basic theory of semi-normal rings, schemes, and rigid analytic spaces is [KL, §1.4, §3.7]. In particular, perfectoid rings are semi-normal and so, for any rigid analytic space X, the pro-étale sheaf $\widehat{\mathcal{O}_X}$ takes values in semi-normal rings; in fact, the pro-étale site of X and of its semi-normalisation are equivalent (as ringed topoi).

   (a) Deduce that if X is not semi-normal, then $\mathcal{O}_X \to R^0\nu_*\widehat{\mathcal{O}_X}$ cannot be an isomorphism. See this explicitly in the case of a cusp $X = \mathrm{Sp}(C\langle X, Y\rangle/(X^2 - Y^3))$ by computing $H^0(X_{proet}, \widehat{\mathcal{O}_X})$. In fact, [KL, Theorem 8.23] proves that $\mathcal{O}_X \to R^0\nu_*\widehat{\mathcal{O}_X}$ is an isomorphism if and only if X is semi-normal; their proof shows how resolutions of singularities enters the picture.

   (b) Are the sheaves $R^j\nu_*\widehat{\mathcal{O}_X}$ coherent? A first attempt might be to try and reduce to the smooth case using resolution of singularities.

   (c) The construction given in the notes still produces a map
$$\Omega^1_{X/C}(-1) \to R^1\nu_*\widehat{\mathcal{O}_X}.$$
   When is this map an isomorphism? Moreover, when is the induced map $\Omega^i_{X/C}(-i) \to R^i\nu_*\widehat{\mathcal{O}_X}$ an isomorphism? For example, is it true with mild control on the singularities of X, such as quotient singularities? Note that if X has quotient singularities (say $X = Y/G$) then the "h-differential forms on X" equal the G-stable forms on Y, by [HJ, Proposition 4.10]. For general X, the case $j = \dim X$ may be most accessible.

[1] This question comes from David Hansen via Kedlaya.

(d) Combining the isomorphism of (a) with [HJ, Proposition 4.5] shows that $R^0\nu_*\widehat{\mathcal{O}_X}$ is related to the h-sheafification of $\mathcal{O}_X$ (here we implicitly assume that X is an algebraic variety, and we abusively also write X for the associated rigid analytic space). Is there a similar relation between $R^j\nu_*\widehat{\mathcal{O}_X}$ and the h-sheaves $\Omega^j_{-/C,h}$ obtained by sheafifying $U \mapsto H^0(U, \Omega^j_{U/C})$ for the h-topology on varieties over C. For example, do we have

$$\dim H^i(X, R^j\nu_*\widehat{\mathcal{O}_X}) = \dim H^i_h(X, \Omega^j_{-/C,h})$$

when X is proper? Note that $H^i_h(X, \Omega^j_{-/C,h})$ is gr^j for Deligne's Hodge filtration on $H^{i+j}_{dR}(X)$.

An alternative approach to some of these questions may come from the notion of sousperfectoid rings. An affinoid algebra R over C is said to be *sousperfectoid* if and only if there exists a perfectoid Tate algebra R_∞ and a continuous algebra homomorphism $R \to R_\infty$ which admits an R-module splitting. It seems to be true that this is equivalent to R being semi-normal[2]. If $R \to R_\infty$ is flat, then sousperfectoid implies semi-normal by [KL, Lemma 1.4.13].

2. **Understanding torsion discrepancies.** Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_C$ with generic fibre X. In this situation, we have several natural integral cohomology theories:

- Étale cohomology $H^n(X_{et}, \mathbf{Z}_p)$.
- Hodge-Tate cohomology $H^n(\widetilde{\theta}^* R\Gamma_A(\mathfrak{X}))$.
- de Rham cohomology $H^n_{dR}(\mathfrak{X}/\mathcal{O}_C)$.
- Crystalline cohomology $H^n_{crys}(\mathfrak{X}_k/W(k))$
- Hodge cohomology $\oplus_{i+j=n} H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/\mathcal{O}_C})$.

Each of these is a finitely presented module over a p-adic valuation ring, and they all have the same rank by fundamental results of p-adic Hodge theory. The first four of these are essentially specializations of $R\Gamma_A(\mathfrak{X})$; the order in which they appear above is roughly inverse to the order in which the corresponding specializations are described in A_{inf}-picture in the notes.

The main theorems of [BMS2], as explained in the notes, imply that the torsion in étale cohomology is bounded above by the torsion in the de Rham and crystalline cohomology. One expects the same relation to hold for Hodge and Hodge-Tate cohomology as well:

(a) Does one have

$$\ell_{\mathbf{Z_p}}(H^n(X_{et}, \mathbf{Z}_p)_{tors}) \leq \ell_{\mathcal{O}_C}(H^i(\widetilde{\theta}^* R\Gamma_A(\mathfrak{X}))_{tors}) \leq \sum_{i+j=n} \ell_{\mathcal{O}_C}(H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/\mathcal{O}_C})_{tors}),$$

where $\ell_{\mathcal{O}_C}$ is the normalized length, as explained in the notes?

[2]This is asserted in problem 6 of http://scripts.mit.edu/~kedlaya/wiki/index.php?title=The_Nonarchimedean_Scottish_Book. It might be worthwhile to rediscover the proof.

As we have seen in the notes, such inequalities can sometimes be strict, and cannot in general be upgraded to an inclusion of torsion subgroups. The goal of this project is to investigate relationships between the torsion subgroups occurring in these cohomology theories, both theoretically as well as through examples. Two natural unanswered questions here are:

(b) By [BMS2], de Rham and Hodge-Tate cohomologies occur as specializations of $R\Gamma_A(\mathfrak{X})$ along θ and $\widetilde{\theta}$. Is there a relation between the torsion subgroups of these cohomology theories? For example, is it always the case that $\ell_{\mathcal{O}_C}(H^n_{dR}(\mathfrak{X}/\mathcal{O}_C)_{tors}) \geq \ell_{\mathcal{O}_C}(H^n(\widetilde{\theta}^* R\Gamma_A(\mathfrak{X}))_{tors})$? In a search for counterexamples, a natural starting point, as in [BMS2, §2], is to construct "interesting" finite flat group schemes over $\mathcal{O}_C$, and to consider cohomology of quotients of smooth projective schemes by free actions of such groups.

(c) Does there exist an example of an $\mathfrak{X}$ as above where the étale and de Rham cohomologies are torsionfree, but the Hodge cohomology is not? What about an example where Hodge cohomology has more torsion than de Rham cohomology?

Notice that we did not include crystalline cohomology above. The reason is that [BMS2, Lemma 4.18] asserts: for a fixed n, $H^n_{crys}(\mathfrak{X}_k/W(k))$ is torsionfree if and only if $H^n_{dR}(\mathfrak{X}/\mathcal{O}_C)$. This is a statement entirely on the "de Rham" side and requires no knowledge of étale cohomology; however, the proof passes through the A_{inf}-cohomology theory and étale cohomology of the generic fibre.

(d) Find a direct proof of the preceding assertion without passing through étale cohomology or the generic fibre.

We end by briefly discussing spectral sequences. The construction of the Hodge-Tate spectral sequence also works integrally to give a spectral sequence

$$E_2^{i,j} : H^i(\mathfrak{X}, \Omega^j_{\mathfrak{X}/\mathcal{O}_C})\{-j\} \Rightarrow H^{i+j}(\widetilde{\theta}^* R\Gamma_A(\mathfrak{X}))$$

converging to the Hodge-Tate cohomology introduced above.

(e) Show that by reduction modulo the maximal ideal of $\mathcal{O}_C$, the integral Hodge-Tate spectral sequence admits a natural map to the conjugate spectral sequence

$$E_2^{i,j} : H^i(\mathfrak{X}_k^{(1)}, \Omega^j_{\mathfrak{X}^{(1)}/\mathcal{O}_C}) \Rightarrow H^{i+j}_{dR}(\mathfrak{X}_k/k),$$

where $\mathfrak{X}_k^{(1)}$ denotes the Frobenius twist relative to k of $\mathfrak{X}$. (This exercise entails understanding the construction of $R\Gamma_A(\mathfrak{X})$.)

(f) Find an $\mathfrak{X}$ as above for which integral Hodge-Tate spectral sequence does not degenerate. In view of the preceding compatibility, a natural starting point would be to find a smooth variety Y/k for which the conjugate spectral sequence does not degenerate, and then find a lift $\mathfrak{X}$ of Y to $\mathcal{O}_C$. Note that the non-degeneration of the conjugate spectral sequence is closely related to the non-liftability of Y to $W_2(k)$ (and thus the non-liftability of $\mathfrak{X}$ to $A_{inf}/\ker(\theta)^2$); this suggests that a suitable Y might be constructed by approximating a finite flat group scheme over k that lifts to $\mathcal{O}_C$ but not to $W_2(k)$.

3. **Perfectoid universal covers for abelian varieties:** Let A/C be an abelian variety. Consider the tower

$$A_\infty := \left(\dots \to A \xrightarrow{p} A \xrightarrow{p} A \right)$$

of multiplication by p maps on A. This tower is an object $A_\infty \in A_{proet}$, and the structure map $f : A_\infty \to A$ is a pro-étale $T_p(A)$-torsor. The question we want to explore is: is A_∞ representable by a perfectoid space? More precisely, is there a perfectoid space that is $\sim$ to (in the sense of [Sc3, Definition 2.20]) the limit of the above tower? In some ways, this question appears to be the p-adic analog of the fact that the universal cover of a complex abelian variety is a Stein space[3].

Why this should be true. When A has good reduction, the arguments sketched in the exercises explain why A_∞ is naturally a perfectoid space. More generally, an affirmative answer in general can likely be extracted in general from a careful reading of [Sc4, §III]. However, any such argument would be necessarily indirect (as it would entail invoking the structure of the boundary in the minimal compactification of $\mathcal{A}_g$, as well as using the Hodge-Tate period map to move the moduli point of A to a "sufficiently close to ordinary" one), and it would be better to come up with a direct argument that is intrinsic to A.

Possible strategy via p-adic uniformization. One might try to construct A_∞ as a perfectoid space by mimicing the construction that works in the good reducition case using the Neron model to replace the non-existent good model, i.e., by contemplating the generic fibre of the p-adically completed inverse limit of multiplication by p on the identity component $\mathcal{A}$ of the Neron model of A. However, this does not quite work: when A has bad reduction, the generic fibre of the p-adic completion of $\mathcal{A}$ is not all of A, but rather just an open subgroup of A (as adic spaces), so at best this approach would construct an open subspace of A_∞ as a perfectoid space. But this suggests an obvious strategy: using p-adic uniformization of abelian varieties, we may write $A = E/M$ in rigid geometry, where E is an extension of an abelian variety B with good reduction by a torus T (and is constructed as an enlargement of the generic fibre of $\mathcal{A}$), and $M \subset T \subset E$ is a lattice of "periods" defining A. In fact, the covering map $\pi : E \to A$ can be constructed from $\mathcal{A}$ (see [BL1, §1] for a summary, and [Hu1, §5] for the adic geometry variant) and has sections locally on A. Thus, one may attempt the following:

(a) Try to show that the inverse limit of multiplication by p on E is naturally a perfectoid space by putting together the analogous assertions for B and T.

(b) If (a) works, then try to conclude that A_∞ is perfectoid using the fact that π has local sections.

Assuming the preceding strategy to represent A_∞ by a perfectoid space works, we would learn:

- Unlike the approach via the Hodge-Tate period map, the approach via p-adic uniformization also potentially applies to "abeloid spaces" A that are not necessarily algebraic (see [Lu]), i.e., the rigid-geometry analog of complex tori; this appears to be the correct generality, at least in analogy with the universal cover from complex geometry.

[3] For example, the perfectoidness of A_∞ implies the following, which can also be seen using the Stein property of the universal cover in complex geometry: for any constructible sheaf F of $\mathbf{F}_p$-vector spaces A, the direct limit $\varinjlim_n H^i(A, [p^n]^* F)$ vanishes for $i > \dim(\mathrm{Supp}(F))$. In other words, the cohomology of constructible $\mathbf{F}_p$-sheaves on A_∞ behaves like that on a Stein space.

- The perfectoidness of A_∞ should yield, via [Sc4, II.2] and almost purity theorem, the following: for any subvariety $X \subset A$, the "universal cover" $X_\infty \to X$ is naturally a perfectoid space. For the more geometrically inclined, it might be fun to try to prove this last statement directly when X is a hyperbolic curve.

4. $L\eta$ **and pro-complexes.** This is essentially a question in homological algebra, but it is motivated by integral p-adic Hodge theory. Fix a complex $K \in D(\mathbf{Z}_p)$ that is derived p-adically complete together with an isomorphism $\phi : L\eta_p(K) \simeq K$. The Berthelot-Ogus theorem [BO2, §8] tells us that the crystalline cohomology complex of any smooth affine scheme in characteristic p carries this structure. What can be said about such K's in general?

   (a) Iterating ϕ gives an isomorphism $L\eta_{p^n}(K) \simeq K$. Proposition 4.3.1 then tells us that K/p^n can be represented by the chain complex $(H^*(K/p^n), \mathrm{Bock}_{p^n})$ for all n. As K is derived p-adically complete, it is tempting to guess that the pair (K, ϕ) carries no homotopical information. More precisely, say $\mathcal{C}$ is the ∞-category[4] of all K as above (suitably defined). Is $\mathcal{C}$ discrete?

   (b) One has the standard restriction map $K/p^{n+1} \to K/p^n$. Via the identification of K/p^n as $(H^*(K/p^n), \mathrm{Bock}_{p^n})$, one can check that this gives a map $R : H^i(K/p^{n+1}) \to H^i(K/p^n)$ on the i-th term that is compatible with the Bockstein differential. On the other hand, there is also a standard map $F : H^i(K/p^{n+1}) \to H^i(K/p^n)$. How are these related? Is there a connection to the F-V-pro-complexes appearing in the work of Langer-Zink (see [LZ], [BMS2, §10.2]).

[4] If you don't know what this means, ask me for a concrete formulation.

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