The Hodge-Tate spectral seq.

<u>Setup</u>: $C/\mathbb{Q}_p$ complete + alg. closed

X/C proper smooth rigid space

<u>Goal</u>: Explain HT ss "by descent from perfectoid spaces", i.e.: construct a map $\nu: (X_{proet}, \hat{\mathcal{O}}_X) \to (X_{et}, \mathcal{O}_{X_{et}})$

s.t the HT ss

$$E_2^{ij}: H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X, \mathbb{Q}_p) \otimes C$$

<u>is</u> the Leray ss for ν

$$E_2^{ij}: H^i(X_{et}, R^j \nu_* \hat{\mathcal{O}}_X) \Rightarrow H^{i+j}(X_{proet}, \hat{\mathcal{O}}_X)$$

Have 3 tasks: (2)

1) Construct ν

2) Show $R^j \nu_* \hat{\mathcal{O}}_X = \Omega^j_{X/\mathbb{C}}(-j)$ ← smoothness

3) Show $H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X) = H^i(X, \mathcal{O}_p) \otimes \mathbb{C}$

properness

Rmk: Conrad-Gabber ⇒ HT ss always degenerates

However: $R\nu_* \hat{\mathcal{O}}_X^+$ is not a direct sum of its cohomology sheaves.

II) The pro-étale site

X smooth adic space $/ C$

Def: $X_{proét} := \left\{ \{U_i\} \in \mathrm{Pro}(X_{ét}) \;\middle|\; \begin{array}{l} U_i \to U_j \text{ finite étale covers} \\ \text{for large } i > j \end{array} \right\}$

$\exists$ a natural notion of coverings.

ex: 1) Any $U \in X_{ét}$ gives you

$U \in X_{proét}$

$\Rightarrow$ get a map $\nu : X_{proét} \longrightarrow X_{ét}$

2) $X = E$ elliptic curve $/C$

$E_\infty := \{ \cdots E \xrightarrow{p} E \xrightarrow{p} E \}$

$\in X_{proét}$

$E_\infty \to E$ is a $\underline{T_p(E)}$-torsor

3) $X = \mathbb{T} = \mathrm{Spa}(\mathbb{C}\langle T^{\pm 1}\rangle, \mathcal{O}_\mathbb{C}\langle T^{\pm 1}\rangle)$

$$X_\infty = \mathbb{T}_\infty = \{ \cdots \mathbb{T} \xrightarrow{(\,)^p} \mathbb{T} \xrightarrow{(\,)^p} \mathbb{T} \}$$

$$\in X_{\text{proet}}$$

$$= \{ \cdots X_{n+1} \longrightarrow X_n \to \cdots \to X_0 \}$$

where $X_n = \mathrm{Spa}(\mathbb{C}\langle T^{\pm \frac{1}{p^n}}\rangle, \mathcal{O}_\mathbb{C}\langle T^{\pm\frac{1}{p^n}}\rangle)$

<u>Note</u>: each $X_{n+1} \longrightarrow X_n$ is a μ_p-torsor

$\Longrightarrow$ $X_\infty \longrightarrow X = X_0$ is a $\underline{\mathbb{Z}_p(1)}$-torsor

<u>Rmk</u>: Given any top space Y, get a sheaf $\underline{Y}$ on X_{proet} via

$$\{U_i\} \longmapsto \mathrm{Map}_{\text{conts}}(\varprojlim U_i, Y)$$

<u>ex</u>: $Y = \mathbb{Q}_p \rightsquigarrow \underline{Y} = \underline{\mathbb{Q}_p}$ <u>Have</u>, $H^i(X_{\text{proet}}, \underline{\mathbb{Q}_p})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^i(X, \mathbb{Q}_p)$

<u>Def</u>: Say $U = \{U_i\} \in X_{proet}$ is <u>affinoid perfectoid</u> if

a) $U_i = \mathrm{Spa}(R_i, R_i^+)$

b) Set $R^+ = (\varinjlim R_i^+)^\wedge$, $R = R^+[\frac{1}{p}]$

Then (R, R^+) is perfectoid.

<u>ex</u>: $X = \mathbb{T} = \mathrm{Spa}(C\langle T^{\pm 1}\rangle, -)$

Then $X_\infty = \mathbb{T}_\infty$ is affinoid perfectoid

$\therefore X_n = \mathrm{Spa}(\cancel{\mathcal{O}_C}\, C\langle T^{\pm 1/p^n}\rangle, -)$

need to check $\Longrightarrow$ $(\varinjlim \mathcal{O}_C\langle T^{\pm \frac{1}{p^n}}\rangle)^\wedge$ is perfectoid

$$\| \qquad\qquad$$

$$\mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle$$

$\therefore X_\infty$ is affinoid perfectoid

(Same thing for more variables)

The $\mathbb{Z}_p(1)$-action on X_∞ is given as (6)

follows:

Given $g \in \mathbb{Z}_p(1)$ corr. to $\underline{\varepsilon} = (1, \varepsilon_p, \varepsilon_{p^2}, \dots)$

$$g \cdot T^{i/p^n} = \varepsilon_{p^n}^{i} \cdot T^{i/p^n}$$

For any $a/p^n \in \mathbb{Z}[\frac{1}{p}]$, write $\varepsilon^i = \varepsilon_{p^n}^{a}$.

Then the action is

$$g \cdot T^i = \varepsilon^i T^i \quad \text{for any}$$

$$i \in \mathbb{Z}[\tfrac{1}{p}]$$

$\therefore$ In the limit, we get:

$$C\langle T^{\pm \frac{1}{p^\infty}} \rangle = \widehat{\bigoplus_{i \in \mathbb{Z}[\frac{1}{p}]}} C \cdot T^i$$

as $\mathbb{Z}_p(1)$-equiv. modules

<u>Thm</u>: $X_{\text{proét}}$ is locally perfectoid:

$\forall\, U \in X_{\text{proét}}$, can find a cover

$$\coprod_i V_i \longrightarrow U \quad \text{s.t}$$

V_i is affinoid perfectoid

Using $\nu : X_{\text{proét}} \longrightarrow X_{\text{ét}}$, get sheaves:

$$\mathcal{O}_X = \nu^{-1}(\mathcal{O}_{X_{\text{ét}}}), \quad \hat{\mathcal{O}}_X = \text{comp. of } \mathcal{O}_X$$
$$= \left(\nu^{-1}(\mathcal{O}^+_{X_{\text{ét}}})\right)^{\wedge}\left[\tfrac{1}{p}\right]$$

<u>ex</u>: Say $X = \mathbb{T}$, $X_\infty \in X_{\text{proét}}$

$$\mathcal{O}_X(X_\infty) = \varinjlim \mathbb{C}\langle T^{\pm \frac{1}{p^n}}\rangle$$

$$\hat{\mathcal{O}}_X(X_\infty) = \mathbb{C}\langle T^{\pm \frac{1}{p^\infty}}\rangle$$

<u>Thm</u>: $\hat{\mathcal{O}}_X$ is acyclic on affinoid perfectoid

(G profinite)

<u>Consequence</u>: Given a $\underline{G}$-torsor $U \to V$

with U affinoid perfectoid, get

$$H^i(V, \hat{\mathcal{O}}_X) = H^i_{cts}(G, \hat{\mathcal{O}}_X(U))$$

<u>Thm</u>: X/C proper $\Rightarrow$

$$H^i(X_{proet}, \hat{\mathcal{O}}_X) \cong H^i(X, \mathbb{Q}_p) \otimes C$$

Pf is a (much harder) version of:

<u>Prop</u>: Say k alg closed of char p

X/k proper

$$H^i(X, \mathbb{F}_p) \cong \{ x \in \underline{\underline{H^i(X, \mathcal{O}_X)}} \mid F(x) = x \}$$

III) Differential forms

X/C smooth of dim d

$$\nu : (X_{\text{proét}}, \hat{\mathcal{O}}_X) \longrightarrow (X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$$

Thm: 1) $R^1\nu_*\hat{\mathcal{O}}_X$ is locally free of rk d

2) $$\Lambda^i R^1\nu_*\hat{\mathcal{O}}_X \cong R^i\nu_*\hat{\mathcal{O}}_X$$

Thm $\approx$ $R^i\nu_*\hat{\mathcal{O}}_X$ has the same size as $\Omega^i_{X/C}(-i)$

Reduce to showing

Prop: $X = \mathbb{T} = \mathrm{Spa}(C\langle T^{\pm 1}\rangle, -)$. Then

$$\Rightarrow H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X) = \begin{cases} \text{free of rk } 1 & i=0, \\ 0 & \text{otherwise} \end{cases}$$

<u>Pf</u>: Consider the $\mathbb{Z}_p(1)$-torsor

$$X_\infty \xrightarrow{\pi} X$$

$$H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X) = H^i_{cts}(\mathbb{Z}_p(1), \hat{\mathcal{O}}_X(X_\infty))$$

$$= H^i_{cts}(\mathbb{Z}_p(1), C\langle T^{\pm 1/p^\infty}\rangle)$$

$$= \widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]}} H^i_{cts}(\mathbb{Z}_p(1), C\cdot T^j)$$

$$= \widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]}} H^i\left(C\cdot T^j \xrightarrow{g-1} C\cdot T^j\right)$$

$$= \widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]}} H^i\left(C\cdot T^j \xrightarrow{\varepsilon^j-1} C\cdot T^j\right)$$

$$= \left(\widehat{\bigoplus_{j\in\mathbb{Z}}} H^i\left(C\cdot T^j \xrightarrow{0} C\cdot T^j\right)\right) \oplus \cancel{\widehat{\bigoplus_{j\in\mathbb{Z}[\frac{1}{p}]-\mathbb{Z}}}} \quad (\text{is } 0)$$

$$g \longleftrightarrow (1, \varepsilon_p, \varepsilon_{p^2}, \dots)$$

$$=$$

$$= \begin{cases} \mathbb{C}\langle T^{\pm 1}\rangle & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$