I) Yesterday:

C complete + alg closed $/\mathbb{Q}_p$

X/C proper smooth rigid-analytic space

$\exists$ an E_2-ss

$$E_2^{ij}: \quad H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}(X, \mathbb{Q}) \otimes C$$

Strategy of proof

1) Construct a "cover" by perfectoid spaces
$\pi: X_\infty \longrightarrow X$,
and study the Hodge coh. of X_∞

2) Descend back down to X

Example :

Say $K | \mathbb{Q}_p$ fin ext , $G_K = Gal(\overline{K}/K)$

Thm : $cd_{\mathbb{F}_p}(G_K) \leq 2$

i.e. $H^i(G_K, M) = 0 \quad \forall i > 2$, M p-torsion

Pf sketch :

Choose $K \xrightarrow{\mathbb{Z}_p} K_\infty \longrightarrow \overline{K}$

perfectoid (after comp.)

FACTS :

1) $cd_{\mathbb{F}_p}(\mathbb{Z}_p) \leq 1$: explicitly calc. !

2) $cd_{\mathbb{F}_p}(K_\infty) \leq 1$: Use the tilting corr. to reduce to :

Prop : R is any $\mathbb{F}_p$-algebra $\Rightarrow$
$H^i(Spec(R)_{et}, \mathbb{F}_p) = 0 \quad \forall i \geq 2$

(1) + (2) + Hochschild-Serre

$\Rightarrow$ Thm.

II) Hodge-Tate decomp. for ell. curves

$K/\mathbb{Q}_p$ finite

$\mathcal{E}/\mathcal{O}_K$ elliptic curve (so: $E = \mathcal{E}_K$ has good red.

$C = \hat{\bar{K}}$.

Goal for today:

·). Construct a G_K-equiv. map

$\alpha: H^0(E, \Omega^1_{E/K}) \to H^1(E_{\bar{K}}, \mathbb{Q}_p) \otimes C(1$

using arithmetic of K.

2) Construct a G_K-equiv. map

$$\cancel{}\ H^1(E, \mathcal{O}_E) \longrightarrow H^1(E_{\overline{K}}, \mathbb{Q}_p) \otimes C$$

using (inspiration from) perfectoid spaces

Background facts

1) $H^1(E_{\overline{K}}, \mathbb{Q}_p) \cong T_p(E_{\overline{K}})^\vee \otimes \mathbb{Q}_p$

where $T_p(E_{\overline{K}}) := \varprojlim_n E(\overline{K})[p^n]$

2) $\mathcal{E}$ satisfies val. criterion

$$\Rightarrow \cancel{}\ \mathcal{E}(\mathcal{O}_C) \cong E(C)$$

3). Elliptic curves are $K(\pi, 1)$'s :

$$H^i(E_{\overline{K}}, \mathbb{Q}_p) \cong H^i_{cts}(T_p(E), \mathbb{Q}_p)$$

4) $[n] : \mathcal{E} \longrightarrow \mathcal{E}$ induces

$$\begin{array}{l} [n]^* : H^i(\mathcal{E}, \mathcal{O}_\mathcal{E}) \longrightarrow H^i(\mathcal{E}, \mathcal{O}_\mathcal{E}) \\ \ \ \| \\ \ n^i \end{array}$$

III) Construction of α :

$$K \subset \bar{K} \subset \hat{\bar{K}} = C$$

$$\mu_{p^\infty} \subset \mathcal{O}_{\bar{K}}^* \xrightarrow{\text{dlog}} \Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}$$

$$f \longmapsto \frac{df}{f}$$

Passing to Tate modules,

$$\mathbb{Z}_p(1) := T_p(\mu_{p^\infty}) \xrightarrow{\text{dlog}} \Omega = T_p(\Omega^1_{\mathcal{O}_{\bar{K}}} \ldots$$

an $\mathcal{O}_C$-module

<u>Thm (Fontaine)</u>: dlog linearizes to

a map

$$\mathcal{O}_C(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \longrightarrow \Omega$$

which is injective, with torsion cokernel

$$\xRightarrow{\text{invert } p} \boxed{C(1) \xrightarrow{\sim} \Omega[\tfrac{1}{p}]}$$

Get a pairing

$$\underset{\substack{\shortparallel \\ E(\bar{K})}}{\mathcal{E}(\mathcal{O}_{\bar{K}})} \times H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/\mathcal{O}_K}) \longrightarrow \Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}$$

$$(x: \mathrm{Spec}(\mathcal{O}_{\bar{K}}) \to \mathcal{E}, \quad \omega) \longmapsto x^*(\omega)$$

<u>Check</u>: this is bilinear

$$
\begin{tikzcd}
H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/\mathcal{O}_K}) \arrow[r] \arrow[d] \arrow[dddddr, bend left] & \mathrm{Hom}(E(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \arrow[d, "\text{apply } T_p(-)"] \\
H^0(E, \Omega^1_{E/K}) \arrow[ddddr, bend left, "\alpha"] & \mathrm{Hom}(T_p(E_{\overline{K}}), \Omega) \arrow[d, phantom, "\cong"] \\
& \underbrace{\mathrm{Hom}(T_p(E_{\overline{K}}), \mathbb{Z}_p)}_{H^1(E_{\overline{K}}, \mathbb{Z}_p)} \otimes_{\mathbb{Q}_p} \Omega \arrow[d, phantom, "\cong"] \\
& H^1(E_{\overline{K}}, \mathbb{Z}_p) \otimes \Omega \arrow[d, "\text{invert } p"] \\
& H^1(E_{\overline{K}}, \mathbb{Q}_p) \otimes \underbrace{\Omega[\tfrac{1}{p}]}_{C(1)} \arrow[d, phantom, "\cong"] \\
& H^1(E_{\overline{K}}, \mathbb{Q}_p) \otimes C(1)
\end{tikzcd}
$$

this is G_K-equiv. by construction

<u>Rmk</u>: This construction makes sense for $\mathbb{G}_m$ as well:

$$\begin{array}{ccc} H^0(\mathbb{G}_m, \Omega^1_{\mathbb{G}_m/\mathbb{Z}_p}) & \longrightarrow & H^1(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes C(1) \\ \rotatebox{90}{$\in$} & & \| \\ \frac{dt}{t} & & \mathbb{Q}_p(-1) \otimes C(1) \\ & & \| \\ & & C \end{array}$$

<u>Exercise</u>: calculate image of $\frac{dt}{t}$

IV) Construction of β

Fix $\mathcal{E}/\mathcal{O}_C$ elliptic curve, $E = \mathcal{E}_C$

<u>Goal</u>: Construct

$$\beta : H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) \longrightarrow H^1(E, \mathbb{Q}_p) \otimes C$$

Consider :

$$\left(\cdots\cdots \mathcal{E} \xrightarrow{[p]} \mathcal{E} \xrightarrow{[p]} \mathcal{E} \right) \Big/ \mathcal{O}_C$$

$\rightsquigarrow$ $\mathcal{E}_\infty$ = limit of $\uparrow$ as schemes $/\mathcal{O}_C$

<u>Rmk</u> : $\mathcal{E}_\infty$ gives a perfectoid space on generic fibres

<u>Obs</u> :

$$\begin{array}{c} \mathcal{E} \xrightarrow{[p^n]} \mathcal{E} \\ \circlearrowleft \\ \mathcal{E}[p^n](\mathcal{O}_C) \\ \| \\ E(C)[p^n] \end{array}$$

$\therefore$ get an action of $T_p(E)$ on $\mathcal{E}_\infty$ that is equiv. for $\mathcal{E}_\infty \xrightarrow{\pi} \mathcal{E}$

bottom $\mathcal{E}$

Have
$$\begin{tikzcd} \mathcal{E}_\infty \arrow[r, "\pi"] \arrow[loop below, "T_p(E)"] & \mathcal{E} \end{tikzcd}$$

Pullback of functions:

$$H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) \longrightarrow H^0(\mathcal{E}_\infty, \mathcal{O}_{\mathcal{E}_\infty})^{T_p(E)}$$

Derive everything:

$$R\Gamma(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) \xrightarrow{\beta_0} R\Gamma_{conts}\left(T_p(E), R\Gamma(\mathcal{E}_\infty, \mathcal{O}_{\mathcal{E}_\infty})\right)$$

Obs: $H^i(\mathcal{E}_\infty, \mathcal{O}_{\mathcal{E}_\infty}) = \varinjlim_{[p]^*} H^i(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$

$$= \begin{cases} H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}) = \mathcal{O}_C & i=0 \\ H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}})[\frac{1}{p}] & i=1 \\ 0 & i>1 \end{cases}$$

derived completions $\Longrightarrow$

$$\widehat{R\Gamma(\mathcal{E}_\infty, \mathcal{O}_{\mathcal{E}_\infty})} \underset{\substack{\uparrow \\ \text{ais}}}{\cong} \mathcal{O}_C[0]$$

$\therefore$ we get:

$$\begin{tikzcd}
R\Gamma(\mathcal{E}, \mathcal{O}_\mathcal{E}) \arrow[r, "\beta_0"] \arrow[dddr, bend right] & R\Gamma_{cts}(T_p(E), R\Gamma(\mathcal{E}_\infty, \mathcal{O}_{\mathcal{E}_\infty})) \arrow[d, "\text{complete}"] \\
& R\Gamma_{cts}(T_p(E), \mathcal{O}_C) \arrow[d, "\wr"] \\
& R\Gamma(E_{et}, \mathcal{O}_C) = \cancel{R\Gamma(E, \mathcal{O}_p)} \arrow[d, equal] \\
& R\Gamma(E_{et}, \mathbb{Z}_p) \otimes \mathcal{O}_C
\end{tikzcd}$$

In deg 1:

$$H^1(\mathcal{E}, \mathcal{O}_\mathcal{E}) \xrightarrow{\beta} H^1(E, \mathbb{Z}_p) \otimes \mathcal{O}_C$$