RATIONAL POINTS ON SURFACES

BIANCA VIRAY
ASSISTANT: ARNE SMEETS
ARIZONA WINTER SCHOOL 2015

1. Outline

Let X be a smooth projective variety over a global field k. This course will focus on obstructions to existence of rational points on X, particularly in the case of surfaces. The rough plan for the lectures is as follows.

1.1. **Lecture 1: Introduction and motivation; The Brauer group.** We will begin by reviewing one or two of the first examples of varieties X that violate the Hasse principle, that is, varieties X that have no k-rational points despite the existence of points over every completion. Then we will define the Brauer group and explain how the examples can be viewed in terms of the Brauer-Manin obstruction. We will also define the étale-Brauer obstruction and say a few words about the limitations of these obstructions.

1.2. **Lecture 2: Classification of surfaces with a view to arithmetic properties and expectations.** We will give a brief overview of the classification of surfaces. For a number of classes of surfaces, we will discuss what is known about their Brauer group, which obstructions are necessary or insufficient, and which obstructions are (conjecturally) sufficient.

1.3. **Lectures 3 and 4: Examples.** In the last two lectures, we will go through one or two examples in detail, explaining how to compute their Brauer group, the Brauer-Manin set and the étale-Brauer set.

2. Project descriptions

In the following projects, X, Y, and S will denote smooth projective geometrically integral varieties over a field k. We will write $\overline{k}$ for the separable closure of k, and $\overline{X}, \overline{Y}, \overline{S}$ for the base change of X, Y, and S respectively to $\overline{k}$. Additionally G_k will denote the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For any regular variety V, we write the Brauer group of V, denoted $\mathrm{Br}\, V$, for the cohomology group $\mathrm{H}^2_{\mathrm{et}}(V, \mathbb{G}_m)$; we abbreviate $\mathrm{Br}\,\mathrm{Spec}\, A$ by $\mathrm{Br}\, A$. If V is a k-scheme, then $\mathrm{Br}_1 V := \ker(\mathrm{Br}\, V \to \mathrm{Br}\, \overline{V})$.

2.1. **Double covers of surfaces and 2-torsion Brauer classes.** Let $\pi\colon X \to S$ be a smooth double cover of a rational geometrically ruled surface. If k is separably closed and has characteristic different from 2, the pullback map $\pi^*\colon \mathrm{Br}\,\mathbf{k}(S) \to \mathrm{Br}\,\mathbf{k}(X)$ surjects onto the subgroup $\mathrm{Br}\, X[2]$ [CV14]. Since every non-constant Brauer class on S is ramified somewhere, this means that we may study unramified 2-torsion Brauer classes on X by studying ramified Brauer classes on S, which is generally easier. The proof relies heavily on the separably closed hypothesis. If the ground field is *not* separably closed, does $\operatorname{im}\pi^*$ still contain $\operatorname{Br} X[2]$?

Possible approaches:

- Kresch and Tschinkel have computed the Brauer group of certain diagonal degree 2 del Pezzo surfaces [KT04]. Since any degree 2 del Pezzo surface is a double cover of $\mathbb{P}^2$, you could first test whether π^* surjects onto $\operatorname{Br} X[2]$ in these examples.
- You could consider the possibly simpler question of whether $\operatorname{Br}_1 X[2]$ is contained in $\operatorname{im}\pi^*$.

2.2. **Relationship between** $\operatorname{Br} Y$ **and** $\operatorname{Br} Y^\tau$ **where** Y^τ **is a twist of** Y**.** If k is a global field, then the étale-Brauer set of X is

$$X(\mathbb{A}_k)^{\mathrm{et,Br}} := \bigcap_{\substack{G \text{ finite étale} \\ f\colon Y \to X, \\ \text{a } G-\text{torsor}}} \bigcup_{[\tau]\in \mathrm{H}^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\operatorname{Br} Y^\tau}).$$

To aid in computation of $X(\mathbb{A}_k)^{\mathrm{et,Br}}$, it would be desirable to determine the strongest possible relationship between $\operatorname{Br} Y$ and $\operatorname{Br} Y^\tau$. You might start with considering the algebraic Brauer group and trying to understand how the action of G_k on $\operatorname{Pic}\overline{Y}$ differs from the action of G_k on $\operatorname{Pic}\overline{Y}^\tau$.

2.3. **Central simple algebra representatives for** p**-torsion transcendental Brauer classes with** $p > 2$**.** Let k be a separably closed field of characteristic different from p, and let $\pi\colon X \to \mathbb{P}^2$ be a p-cyclic cover over k. Then there is an action of $\mathbb{Z}[\zeta]$ on $\operatorname{Br} X$ and $\operatorname{Br} X[1-\zeta] \subset \operatorname{im}\pi^*\colon \operatorname{Br} U \to \operatorname{Br} \mathbf{k}(X)$, where U is the open set of $\mathbb{P}^2$ obtained by removing the branch curve of π and a fixed line [IOOV]. The results and proofs of [CV14] suggest a construction of central simple algebras on $\mathbb{P}^2$ with prescribed ramification. If successful, this together with the results from [IOOV] would give central simple algebra representatives for elements of $\operatorname{Br} X[1-\zeta]$.

2.4. **The cokernel of** $\operatorname{Br}_1 X \to \mathrm{H}^1(G_k, \operatorname{Pic}\overline{X})$**.** By the Hochschild-Serre spectral sequence, we have an exact sequence

$$\operatorname{Br}_1 X \to \mathrm{H}^1(G_k, \operatorname{Pic}\overline{X}) \to \mathrm{H}^3(G_k,).$$

If k is a global field, then $\mathrm{H}^3(G_k, \mathbb{G}_m) = 0$ so every element of $\mathrm{H}^1(G_k, \operatorname{Pic}\overline{X})$ lifts to an algebraic Brauer class on X. If k is an arbitrary field, this may no longer hold. For example, Uematsu showed that if $k = \mathbb{Q}(\zeta_3, a, b, c)$ where a, b, c are independent transcendentals, and X is a cubic surface, then the map $\mathrm{H}^1(G_k, \operatorname{Pic}\overline{X}) \to \mathrm{H}^3(k, \mathbb{G}_m)$ can be nonzero [Uem14].

Let X be a del Pezzo surface of degree 4, i.e., a smooth intersection of 2 quadrics in $\mathbb{P}^4$. We may associate to X a pencil of quadrics $V \to \mathbb{P}^1$. A general fiber of V is rank 5; there is a reduced degree 5 subscheme $\mathcal{S} \subset \mathbb{P}^1$ where the quadrics have rank strictly less than 5. There are necessary and sufficient conditions in terms of these quadrics of lower rank for the existence of a nontrivial element of $\mathrm{H}^1(G_k, \operatorname{Pic}\overline{X})$ [VAV14]. If, in addition, certain degenerate quadrics have a rational point (over their field of definition), then there is a construction which lifts a nontrivial element of $\mathrm{H}^1(G_k, \operatorname{Pic}\overline{X})$ to an algebraic Brauer class on X [VAV14]. Determine whether this condition of having a rational point is necessary, i.e., does there exist a field k and a degree 4 del Pezzo surface over k with the map $\mathrm{H}^1(G_k, \operatorname{Pic}\overline{X}) \to \mathrm{H}^3(k, \mathbb{G}_m)$ nontrivial?

2.5. **Brauer groups of del Pezzo surfaces.** Let X be a del Pezzo surface over a global field k. Since X is geometrically rational, $\operatorname{Br} X = \operatorname{Br}_1 X$. Additionally, there are finitely many possibilities for $\operatorname{Br}_1 X/\operatorname{Br} k$ [Cor07, Thm. 4.1]. If we fix the degree of X and assume that X is minimal, i.e., there are no Galois invariant subsets of pairwise skew (-1)-curves, what are the possibilities for $\operatorname{Br}_1 X/\operatorname{Br} k$? Alternatively (or in addition!), for each possible isomorphism class of $\operatorname{Br}_1 X/\operatorname{Br} k$, you could construct a del Pezzo surface X that has that particular Brauer group. For instance, you could try to construct a del Pezzo surface (necessarily of degree 1), with an order 5 Brauer class?

References

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University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA

E-mail address: `bviray@math.washington.edu`

URL: `http://math.washington.edu/~bviray`