Last time: $X(\mathbb{A}_{\mathbb{Q}})^{et,Br} = \emptyset \subseteq \underbrace{X(\mathbb{A}_{\mathbb{Q}})^{Br}}_{?} \subseteq X(\mathbb{A}_{\mathbb{Q}})^{Br_1} \neq \emptyset$

**Thm** (BBMPV)

$$\operatorname{Br} X = \operatorname{Br}_1 X \quad \& \text{ so } \quad X(\mathbb{A}_{\mathbb{Q}})^{Br} \neq \emptyset$$

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General tools for $\operatorname{Br} \overline{X}$ ($X$ smooth var, $\operatorname{char}(k) = 0$)

$$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \to \operatorname{Br} \overline{X} \to H^3(X, \mathbb{Z})_{tors} \to 0$$

$$\frac{\operatorname{Br} X}{\operatorname{Br}_1 X} \hookrightarrow \left(\operatorname{Br} \overline{X}\right)^{G_k}$$

CTS: cokernel is finite

$X$ Enriques: $b_2 = \rho = 10$

$$H^3(X, \mathbb{Z})_{tors} \cong \mathbb{Z}/2$$

Idea: Find a # field $K/\mathbb{Q}$ s.t.

$$Br_1 X_{\cancel{\mathbb{Q}}} \supsetneq Br_0 X_{\cancel{\mathbb{Q}}} \quad \& \quad \left(\frac{Br X_K}{Br_0 X_K}\right)^{Gal(K/\mathbb{Q})} \subseteq \frac{Br_1 X_K}{Br_0 X_K}$$

How do we write down Brauer classes &

given $\alpha \in Br$, how do ~~we~~ det. if $\alpha - \sigma(\alpha)$ is trivial?

Recall from Br(fields):

* Cyclic algebras : $L/k$ is a cyclic extn

then $\mathrm{Br}(L/k) \simeq k^\times / N(L^\times)$

$(\sigma, b) \longleftarrow\!\shortmid b$

$\sigma$ fixed gen $L/k$

* Purity thm for smooth, proj. rat'l varieties $S$ / char 0 field

$$0 \to \mathrm{Br}_0 S \longrightarrow \mathrm{Br}\, k(S) \xrightarrow{\partial_s} \bigoplus_{s \in S^{(1)}} H^1(\kappa(s), \mathbb{Q}/\mathbb{Z})$$

th

<u>Setup</u> $S$ smooth proper rat'l g. ruled

& $\pi : Y \to S$ smooth proper double cover

1) $S$ rat'l $\quad \dfrac{\operatorname{Br} k(S)}{\operatorname{Br}_0 S} \hookrightarrow \displaystyle\bigoplus_{s \in S^{(1)}} H^1(k(s), \mathbb{Q}/\mathbb{Z})$

2) $k(Y)/k(S)$ cyclic ext'n so

$\ker \pi^* : \operatorname{Br} k(S) \to \operatorname{Br} k(Y)$

understood

$\rightsquigarrow$ understand $\operatorname{im} \pi^* \cap \operatorname{Br} Y$ through residues on $S$

$$\downarrow$$

$$\operatorname{Br} \overline{Y}[2] \subseteq \operatorname{im} \pi^* \cap \operatorname{Br} \overline{Y}$$

~~[illegible]~~ Creutz.V
Ingalls, Obus, Ozman, V
BBMPV

Restrict to $\mathbb{P}^1 \times \mathbb{P}^1 \supseteq B$ branch locus of $\pi$
smooth, geom. irred.

<u>Prop1</u> Let $\alpha \in \mathrm{Br}\, k(S)$ be s.t. $\pi^*\alpha \in \mathrm{Br}\, \pi^{-1}(\mathbb{A}^2)$
Then $\exists \beta \in \underline{\mathrm{Br}\, \mathbb{A}^2 \setminus B}$ s.t. $\pi^*\alpha = \pi^*\beta$

$$\rightsquigarrow \frac{\mathrm{Br}\, \mathbb{A}^2 \setminus B}{\mathrm{Br}\, k} \overset{\partial_B}{\hookrightarrow} H^1(\kappa(B), \mathbb{Q}/\mathbb{Z})$$

<u>Prop2</u> Let $\alpha \in \mathrm{Br}\, \mathbb{A}^2 \setminus B$ s.t. $\pi^*\alpha \in \mathrm{Br}_0\, Y$
Then $\exists\, D \subseteq Y \setminus \pi^{-1}(B)$ s.t. (div.)

$$\partial_B(\alpha) = [g|_B] \qquad g \in k(S)^\times$$

where $\mathrm{div}(g) = \pi_* D - m(\mathbb{P}^1 \times \infty) - n(\infty \times \mathbb{P}^1)$

$$\pi_* : \operatorname{Div} Y \longrightarrow \operatorname{Div} S$$

$$Z \longmapsto \underbrace{\qquad}_{[k(Z)\,:\,k(\pi(Z))]}\pi(Z)$$

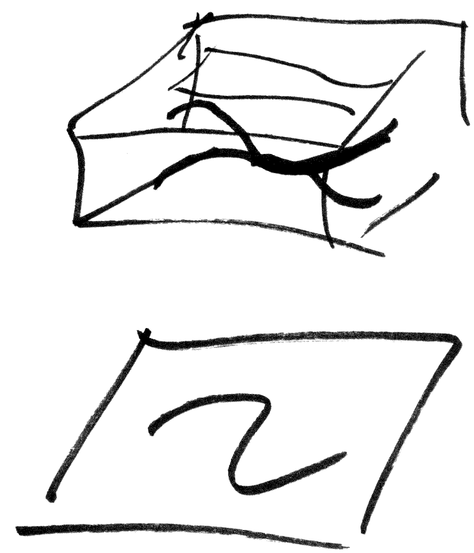
If $n_i \equiv 0 \bmod 2$, then OK

$$\tilde{Z}_i = \frac{n_i}{2}\,\pi^{-1}(Z_i)$$

If $n_i \equiv 1 \bmod 2$,

then I need

$$\boxed{\pi^{-1}(Z_i) = \tilde{Z}_i \cup \tilde{Z}_i'}$$



For simplicity $k = \bar{k}$

$\alpha \in \mathrm{Br}\, \mathbb{A}^2 \setminus B$ $\quad \alpha \in \ker \pi^*$

$\alpha \overset{\Downarrow}{=} (f, g)_{-1}$ where $k(S)(\sqrt{f}) = k(Y)$

& $g \in k(S)^\times$

$= (f, g(-f)^i)_{-1} = (g(-f)^i, f)$

so WMA $B \notin \mathrm{supp}(g)$

Then $\partial_B(\alpha) = [g|_B]$

$$\pi_* D - m(\mathbb{P}^1 \times \infty) - n(\infty \times \mathbb{P}^1)$$

$$\operatorname{div}(g) = \underbrace{\sum n_i Z_i - m(\mathbb{P}^1 \times \infty) - n(\infty \times \mathbb{P}^1)}_{\parallel}$$

<u>Want</u> $n_i Z_i = \pi_* (\widetilde{Z_i})$ for $Z_i \subseteq Y$

$\forall s \in S^{(1)} \setminus \{\mathbb{P}^1 \times \infty, \infty \times \mathbb{P}^1, B\} = (\mathbb{A}^2 \setminus B)^{(1)}$

want either
- $v_s(g) \equiv 0 \bmod 2$
- or $s$ splits comp. in $k(Y)/k(s)$

$\Longleftrightarrow$

$\partial_s(g\alpha) = 0$
$\forall s \in (\mathbb{A}^2 \setminus B)^{(1)}$ $\Longleftrightarrow$ $\alpha \in \mathrm{Br}\, \mathbb{A}^2 \setminus B$

For any ext'n $K/k$

$$0 \to \frac{\operatorname{Pic} Y_K}{\pi^* \operatorname{Pic} B_K + 2 \operatorname{Pic} Y_K} \to \left( \frac{\operatorname{Pic} B_K}{\langle \mathbb{P}^1 \times \infty, \infty \times \mathbb{P}^1 \rangle} \right)[2] \to \frac{\operatorname{Br} \pi^{-1}(\mathbb{A}^2_K)}{\operatorname{Br}_0 \pi^{-1}(\mathbb{A}^2_K)}[2]$$

deg 2 K3 $Y \to \mathbb{P}^2$

$\mathbb{P}^2 \times \mathbb{P}^2$ deg 8 $\longleftrightarrow$ 2-tor in Jac

$\longleftrightarrow$ even theta

cubic 4fold $\longleftrightarrow$ odd theta