d sqfree integer

$$Y^d: \quad \begin{aligned} ds^2 &= xy + 5z^2 && Q_1 \\ d(s^2 - 5t^2) &= x^2 + 3xy + 2y^2 && Q_2 \\ du^2 &= 12x^2 + 111y^2 + 13z^2 && Q_3 \end{aligned} \quad \subseteq \mathbb{P}^5$$

deg 8
K3

$$\sigma : \mathbb{P}^5 \longrightarrow \mathbb{P}^5$$

$$[s, t, u, x, y, z] \longmapsto [-s, -t, -u, x, y, z]$$

$\sigma|_Y$ has no fixed pts

$X = Y/\sigma$ is an Enriques surface

Thm (VAV, BBMPV)

$$\underline{X(\mathbb{A}_{\mathbb{Q}})^{et, Br} = \emptyset} \quad \& \quad X(\mathbb{A}_{\mathbb{Q}})^{Br} \neq \emptyset$$

$$X(\mathbb{A}_{\mathbb{Q}})^{Br} = X(\mathbb{A}_{\mathbb{Q}})^{Br_1}, \quad \emptyset \neq X(\mathbb{A}_{\mathbb{Q}})^{Br_1}$$

Prop $Y(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ hence $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$

Pf Weil conj: $V/\mathbb{F}_q$ smooth proj, red. of $\tilde{V}/\bar{\mathbb{Q}}$

then $$\# V(\mathbb{F}_{q^m}) = \sum_{i=0}^{2n} (-1)^i \sum_{j=1}^{b_i(\tilde{V})} \alpha_{ij}^m$$

$\alpha_{ij} \in \bar{\mathbb{Z}}$

$|\alpha_{ij}| = q^{i/2}$

Y K3 surface : betti #s 1, 0, 22, 0, 1

$$\# Y(\mathbb{F}_p) = 1 + \sum_{j=1}^{22} \alpha_{2j} + p^2$$

$$\geq 1 - 22p + p^2$$

$\rightsquigarrow Y(\mathbb{F}_p) \neq \emptyset$ if $p \geq 23$

Hensel's Lemma $\Big\}$ Y has good red at $\mathbb{F}_p$

$\downarrow$

$Y(\mathbb{Q}_p) \neq \emptyset$

left with

2, 3, 5, 13, 37, 59, 151, 157, 179, 821,
881, 1433

Computer says yes.

Prop $X(\mathbb{A}_{\mathbb{Q}})^{et,Br} = \emptyset$

Recall

$$X(\mathbb{A}_{\mathbb{Q}})^{et,Br} = \bigcap_{\substack{f':Y'\to X \\ G' \text{ finite} \\ \text{etale}}} \bigcup_{[\tau]\in H^1(k,G)} f'^{\tau}\left(Y'^{\tau}(\mathbb{A}_{\mathbb{Q}})^{Br}\right)$$

X Enriques $\Rightarrow$ only nontriv étale cover is K3.

$$X(\mathbb{A}_{\mathbb{Q}})^{et,Br} = \bigcup_{\substack{H^1(\mathbb{Q},\mathbb{Z}/2) \\ \text{||} \\ d\in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}}} f^d\left(Y^d(\mathbb{A}_{\mathbb{Q}})^{Br}\right) \cap X(\mathbb{A}_{\mathbb{Q}})^{Br}$$

Case 1 $\exists p | d$, $p \nmid 10$ then $Y^d(\mathbb{Q}_p) = \emptyset$

Assume that there is a sol'n

Case 1a: $x \equiv y \equiv z \equiv 0 \bmod p$ $\Rightarrow\Leftarrow$

Case 1b: at least one of x, y, z nonzero mod p

$$xy + 5z^2 \equiv 0 \bmod p$$

$$12x^2 + 111y^2 + 13z^2 \equiv 0 \bmod p$$

$\swarrow$ $\searrow$

$x + y \equiv 0 \bmod p$ $\qquad$ $x + 2y \equiv 0 \bmod p$

$\downarrow$

$x \equiv -y \bmod p$

$-x^2 + 5z^2 \equiv 0$ $\qquad$ $\Rightarrow\Leftarrow$ unless $p \equiv 157$

$123x^2 + 13z^2 \equiv 0$ $\qquad$ $\Downarrow$

$5 \notin \mathbb{F}_p^{\times 2}$ ✓

Case 2 $\quad d<0 \quad Y^d(\mathbb{R})=\emptyset$

Case 3 $\quad d \equiv 2\,(3) \quad Y^d(\mathbb{Q}_3)=\emptyset$

Case 4 $\quad d \underset{m}{\equiv} 10 \quad Y^{10}(\mathbb{Q}_5)=\emptyset$

$\rightsquigarrow d=1$

$\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$

$$g: Y \longrightarrow S^4$$

$$s^2 = xy + 5z^2$$

$$(s^2-5t^2) = (x+y)(x+2y)$$

$$\alpha = \left(5, \frac{x+y}{x}\right) \in \operatorname{Br} S$$

Birch & Swinnerton-Dyer

$$S(\mathbb{A}_\mathbb{Q})^\alpha = \emptyset \Rightarrow S(\mathbb{Q}) = \emptyset$$

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{A}_\mathbb{Q} \arrow[r] \arrow[dr] & Y \arrow[d] & \\
& S & \alpha \in \operatorname{Br} S \arrow[ull, bend right]
\end{tikzcd}$$

$$g\left(Y(\mathbb{A}_\mathbb{Q})^{g^*\alpha}\right) \subseteq S(\mathbb{A}_\mathbb{Q})^\alpha$$

Algebraic Brauer set

Recall $\dfrac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \xrightarrow{\sim} H^1(G_{\mathbb{Q}}, \mathrm{Pic}\,\overline{X})$

For any Enriques surface

$$\begin{array}{ccccccccc} & & & & & & \mathrm{NS}(\overline{X}) & & \\ & & & & & & \parallel & & \\ 0 & \to & \langle K_X \rangle & \to & \mathrm{Pic}\,\overline{X} & \to & \mathrm{Num}(\overline{X}) & \to & 0 \\ & & \wr\wr & & & & \wr\wr & & \\ & & \mathbb{Z}/2 & & & & U \oplus E_8(-1) & & \end{array}$$

$$\begin{array}{ccccccc} H^0(\mathrm{Num}\,\overline{X}) & \to & H^1(G_{\mathbb{Q}}, \langle K_X \rangle) & \to & H^1(G_{\mathbb{Q}}, \mathrm{Pic}\,\overline{X}) & \to & H^1(G_{\mathbb{Q}}, \mathrm{Num}\,\overline{X}) \to \cdots \\ & & \wr\wr & & & & \\ & & \mathbb{Q}^\times/\mathbb{Q}^{\times 2} & & & & \end{array}$$

<u>Prop</u>

$$\cancel{\text{HI}}\quad X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{im} H^1(\langle K_X \rangle)} = \bigcup_{d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}} f^d\left(Y^d(\mathbb{A}_{\mathbb{Q}})\right)$$

$$\cancel{\phi} \quad \leftarrow \quad Y^1(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$$

What is $H^1(\operatorname{Num}\bar{X})$?

Need to know $\operatorname{Num}\bar{X}$ as a Galois module.

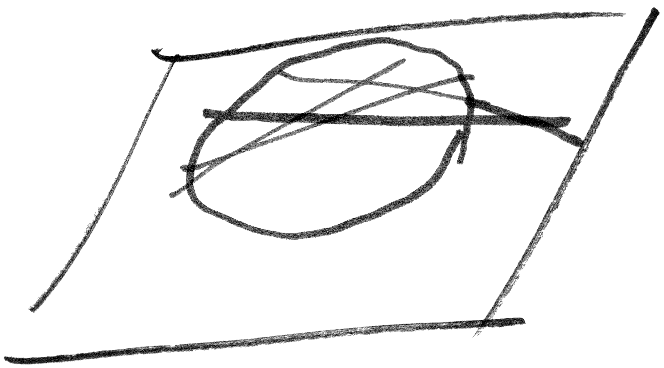
Find curves on $\bar{X}$!

$\rightsquigarrow$ Find curves on $\bar{Y}$ whose classes are σ-inv!

$Y \rightsquigarrow \mathcal{Y} \longrightarrow \mathbb{P}^2$ net of quadrics

$\mathcal{Y} \subseteq \mathbb{P}^5 \times \mathbb{P}^2$

$\lambda Q_1 + \mu Q_2 + \nu Q_3$



degeneracy locus of net

a isol. sing pt of deg locus

$\wr$

rk 4 quadric

14 sings pts

F_i, G_i

$i = 1, \dots, 14$

$F_i + G_i = F_j + G_j$

Let Q_T be a rk 4 quadric

$$Y \subseteq V(Q_T)$$

$$\overset{\shortparallel}{V(Q_T, Q, Q')}$$

$$\begin{tikzcd} & V(Q_T) \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

genus 1 fibrations