X smooth proper variety / global field k

$$X(k) \subseteq X(\mathbb{A}_k)^{et,Br} \subseteq X(\mathbb{A}_k)^{Br} \subseteq X(\mathbb{A}_k)$$

$$X(\mathbb{A}_k)^{Br} \subseteq X(\mathbb{A}_k)^{Br_1 \cup /}$$

<u>Q</u> Determine^{a} class of varieties S s.t.

- $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$ satisfy HP
- $X(\mathbb{A}_k)^{Br} \neq \emptyset \Rightarrow X(k) \neq \emptyset$ BM is only obs
- $X(\mathbb{A}_k)^{et,Br} \neq \emptyset \Rightarrow X(k) \neq \emptyset$ ét-Br is only obs

<u>Ideally</u> S would be described geometrically.

<u>Ex</u> $S = \{$quadric hypersurfaces$\}$ HP holds for S

$S = \{$genus 1 curves$\}$ Manin: assuming $\#Ш(E) < \infty$
$\Rightarrow$ BM is only obs for S

Thm (CTPS)

X, Y smooth proper k-birat'l   If $k = \#$field

$$X(\mathbb{A}_k)^{Br} \neq \emptyset \iff Y(\mathbb{A}_k)^{Br} \neq \emptyset$$

$$X(\mathbb{A}_k)^{et, Br} \neq \emptyset \iff Y(\mathbb{A}_k)^{et, Br} \neq \emptyset$$

Classification of smooth proj. surfaces

$\mathrm{char}(k) = 0$

Kodaira dimension ω_X canonical sheaf

Consider

$$\phi_{\omega_X^{\otimes n}} : X \longrightarrow \mathbb{P}(H^0(X, \omega_X^{\otimes n}))$$

$$\kappa(X) := \begin{cases} \max\limits_{n>0} \dim \phi_{\omega_X^{\otimes n}}(X) & \text{for all } n \gg 0 \ H^0(X, \omega_X^{\otimes n}) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

<u>Lemma</u> (Lang-Nishimura)

$f: X \dashrightarrow Y$ rat'l map of k-schemes

If Y proper & X ~~smooth~~ has a smooth k-pt

then Y has a k-pt

<u>Pf</u> Induction on $\dim X =: n$

$n=0$ ✓

$n>0$

$$\begin{tikzcd} X' = Bl_p X \arrow[r] \arrow[rr, dashed, bend right] & X \arrow[r, dashed] & Y \end{tikzcd}$$

Y proper $\Rightarrow$ defined outside of codim ≥ 2 set

$\rightsquigarrow E \dashrightarrow Y$ rat'l map

$\mathbb{P}^{n-1} \cong E$

$$\kappa(X) \in \{-\infty\} \cup \{0, \dots, \dim(X)\}$$

$\dim(X) = 1$

$\kappa(X)$	$-\infty$	0	1
g	0	1	≥ 2

Def If $\kappa(X) = \dim X$, then X is general type

Lang conj If X general type then $X(k)$ are not Zariski dense

Sarnak-Wang '95

If Lang conj holds then $\exists$ smooth hypersurface X $\dim \geq 3$ s.t $X(\mathbb{A}_k)^{Br} \neq \emptyset$ & $X(k) = \emptyset$

Surfaces of neg. Kodaira dim.

Thm (Enriques)
If X smooth proj surface w/ $\kappa(X) < 0$ then $\overline{X}$ is ruled, i.e. $\exists$ smooth proj. C & $f: \overline{X} \dashrightarrow C$ s.t. $\overline{X}_\eta \simeq \mathbb{P}^1_{k(C)}$

Prop $\kappa(X) < 0$ & $\overline{X}$ ruled over a positive genus curve then $\exists$ smooth proj curve $C/_k$ $g(C) > 0$ & $f: X \longrightarrow C$ over k s.t. X_η is genus 0, i.e. X is a conic bundle over a positive genus curve.

Thm (Iskovskikh) If $\overline{X}$ is rat'l then either
- X is birat'l to a conic bundle / genus 0 curve

or
- ω_X^{-1} is ample, i.e. X is del Pezzo surface.

Rmks deg(del Pezzo) = $\omega_X \cdot \omega_X \in [1,\dots,9]$

Conj (Colliot-Thélène, Sansuc)

BM is the only obs for geom. rat'l surfaces

$\parallel$

alg BM

Known: · for dP's deg ≥ 5

· for some rat'l conic bundles

· cond. on $Ш(E) < \infty$ & Schinzel^2 for some dP4

Surfaces of $\kappa = 0$

Let X be a minimal surface w/ $\kappa(X) = 0$. Then X is

- a twist of an abelian surface
- a bielliptic surface $\overline{X} \simeq E_1 \times E_2 / G$ $\quad G \subseteq E_{1,\mathrm{tors}}$, $E_2/G \simeq \mathbb{P}^1$
- a K3 surface
- an Enriques surface, i.e. $h^1(X, \mathcal{O}_X) = h^0(X, \omega_X) = 0$

Thm (Manin + ε)

If $\# Ш(A) < \infty$ for all abel. surfaces then

- BM is only obs for twists of abel. surfaces
- ét-Br is only obs for bielliptic surfaces

Skor '99: $\exists$ bielliptic surf w/ $X(\mathbb{A}_k)^{et, Br} = \emptyset$ & $X(\mathbb{A}_k)^{Br} \neq \emptyset$

Thm Let Y be a K3 surface & $\sigma : Y \to Y$ a fixed pt free invol. Then $X := Y/\sigma$ is an Enriques surface.

Conversely, If X Enriques, $\exists$ a K3 Y & ~~$\sigma : Y \to Y$ fpf inv. s.t.~~ $f : Y \to X$ étale double cover

Conj (Skorobogatov)

BM is only obs for K3 surfaces

$\Downarrow$

ét Br is only obs for Enriques surfaces.

X surface / # field

$$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \to \operatorname{Br} \overline{X} \to H^3(X, \mathbb{Z})_{tors} \to 0$$

Thm (SZ)

X K3 surface/# field

$$\frac{\mathrm{Br}\,X}{\mathrm{Br}_0 X} \text{ is finite}$$

$$0 \to \frac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \to \frac{\mathrm{Br}\,X}{\mathrm{Br}_0 X} \to \left(\mathrm{Br}\,\overline{X}\right)^{\Gamma}$$