k global field

X smooth geom. integral k-variety

Ω_k = set of places of k

$v \in \Omega_k$ $\quad k_v$ completion, $\mathcal{O}_v$ val ring (if v nonarch)

$\mathbb{A}_k$ - adèle ring of k

<u>Q</u> Given X/k how do we det if $X(k) \neq \emptyset$?

If $X(k) \neq \emptyset$ $\quad$ then $X(k_v) \neq \emptyset$

$\qquad$ then $\overbrace{\prod_{v\in\Omega_k}{}^{'}\left(X(k_v), X(\mathcal{O}_v)\right)}^{X(\mathbb{A}_k)\ =} \neq \emptyset$

Rmk X proper $\quad X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$

k_v is complete so $X(k_v)$ is easier to compute

Furthermore, Weil conj. show that $X(k_v) \neq \emptyset$

$$\forall v \notin \underset{\substack{\uparrow \\ \text{finite}}}{S} \subseteq \Omega_k$$

The Brauer-Manin obstruction

Given a field F & an F-pt: $P: \operatorname{Spec} F \to X$

we obtain $P^*: \operatorname{Br} X \longrightarrow \operatorname{Br} F$

$$\operatorname{Br} X = H^2(X, \mathbb{G}_m) \subseteq \operatorname{Br} k(X)$$

Fix $\alpha \in \operatorname{Br} X$

$$\alpha(P) = P^* \alpha$$

Example $\quad X: \begin{array}{l} xy + 5z^2 - s^2 = 0 \\ (x+y)(x+2y) - s^2 + 5t^2 = 0 \end{array} \subseteq \mathbb{P}^4$

$\alpha := \left(5, \frac{x+y}{x}\right)_{-1}$ Claim $\alpha \in \operatorname{Br} X$

$x \in X^{(1)}$

$\partial_x(\alpha) = 0 \iff$ v_x splits completely in $k(X)\sqrt{5}$

or $v_x\left(\frac{x+y}{x}\right) \equiv 0 \bmod 2$

If $x \leftrightarrow V(x+y)$
then in $k(X)\sqrt{5}$ x splits completely

What is $\alpha(P)$?
If $P \notin V(x+y) \cup V(x)$ then $\alpha(P) = \left(5, \frac{x+y}{x}(P)\right)$

Claim

$$\alpha(P_v) = \begin{cases} 0 & \text{if } v \neq 5 \\ 1/2 & \text{if } v = 5 \end{cases}$$

Fix v
$\forall P_v \in X(\mathbb{Q}_v)$

$\Rightarrow X(\mathbb{Q}) = \emptyset$

Fix $\alpha \in \operatorname{Br} X$

$$
\begin{tikzcd}
 & X(k) \arrow[r, hook] \arrow[d, "\alpha(-)"'] & X(\mathbb{A}_k) \arrow[d, "\alpha(-)"'] \arrow[ddr, bend left, "\varphi_\alpha"] & & \\
 & \operatorname{Br} k \arrow[d, equal] & \prod{}'(\operatorname{Br} k_v, \operatorname{Br} \mathcal{O}_v) \arrow[d, equal] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r, hook] & \bigoplus \operatorname{Br} k_v \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

complex $\implies X(k) \subseteq \varphi_\alpha^{-1}(0) =: X(\mathbb{A}_k)^{\alpha}$

For any $S \subseteq \operatorname{Br} X$

$$X(k) \subseteq X(\mathbb{A}_k)^{S} = \bigcap_{\alpha \in S} X(\mathbb{A}_k)^{\alpha}$$

$S = \operatorname{Br} X$ $\quad X(\mathbb{A}_k)^{\operatorname{Br}} = X(\mathbb{A}_k)^{\operatorname{Br} X}$

Lemma If X smooth, then

$$\alpha(-): X(k_v) \longrightarrow \mathrm{Br}\, k_v$$ locally constant.

Rmk X/k

$$\mathrm{Br}_0 X = \mathrm{im}(\mathrm{Br}\, k \longrightarrow \mathrm{Br}\, X)$$

$$\boxed{X(\mathbb{A}_k)^{\mathrm{Br}_0 X} = X(\mathbb{A}_k)}$$

$$\rotatebox{90}{$\supseteq$}$$

$$\mathrm{Br}_1 X = \ker(\mathrm{Br}\, X \to \mathrm{Br}\, X^{\mathrm{sep}})$$

$\rotatebox{90}{$\supseteq$}$ algebraic elements

$$\mathrm{Br}\, X$$

$$\begin{tikzcd} \mathrm{Spec}\, F \arrow[r] \arrow[dr, dashed] & X \arrow[d] \\ & \mathrm{Spec}\, k \end{tikzcd}$$

$$X(\mathbb{A}_k)^{\mathrm{Br}_1}$$

1999 Skorobogatov defined étale-Brauer obstruction

G finite étale grp scheme $/ k$

$f: Y \to X$ ~~fppf~~ étale G-torsor

Let $x \in X(k)$ then $f_x : Y_x \to x$ G-torsor $/k$

$$X(k) = \coprod_{[\tau] \in H^1(k,G)} \underbrace{\left\{ x \in X(k) : [Y_x] = [\tau] \right\}}_{} \quad \left(X^{\cancel{\tau}}(k)\right)^{\tau}$$

For τ a cocycle repr an elt in $H^1(k,G)$, can const.

$f^{\tau} : Y^{\tau} \longrightarrow X$ G^{τ}-torsor s.t.

$$f^{\tau}\left(Y^{\tau}(k)\right) = \left(X(k)\right)^{\tau}$$

$$X(k) = \coprod_{[\tau] \in H^1(k,G)} f^\tau(Y^\tau(k))$$

$$\begin{array}{ccc} \cap\!\!\! \mid\mid & & \mid\mid\!\cap \end{array}$$

$$X(\mathbb{A}_k)^{et,Br} := \bigcap_{\substack{f: Y \to X \\ G \text{ finite} \\ \text{étale}}} \coprod_{[\tau] \in H^1(k,G)} f^\tau\left(Y^\tau(\mathbb{A}_k)^{Br}\right) \subseteq X(\mathbb{A}_k)$$

$f: Y \to X$

Prop If X proper then $Y^\tau(\mathbb{A}_k) = \emptyset$ for all $\tau \in H^1(k,G)$ outside of a finite set.

Ex

$$Y^{(d)}: \begin{array}{l} d(xy+5z^2)-s^2=0 \\ d((x+y)(x+2y))-s^2+5t^2=0 \\ d(12x^2+111y^2+13z^2)-u^2=0 \end{array}$$

$\sigma : \mathbb{P}^5 \longrightarrow \mathbb{P}^5$ $\quad$ $\sigma|_Y$ has no fixed pts

$(s,t,u,x,y,z) \longmapsto (-s,-t,-u,x,y,z)$

so $f: Y \longrightarrow X = Y/\sigma$ is an étale torsor under $\mathbb{Z}/2$

$H^1(\mathbb{Q}, \mathbb{Z}/2) \cong \mathbb{Q}^\times / \mathbb{Q}^{\times 2} \ni [d]$ $\quad$ d sqfree int.

$\cancel{Y^{(d)}} =$

$$X(k) \subseteq X(\mathbb{A}_k)^{et,Br} \subseteq X(\mathbb{A}_k)^{Br} \subseteq X(\mathbb{A}_k)^{Br_1} \subseteq X(\mathbb{A}_k)$$

<u>Thm</u> (Poonen '09) $\forall$ ~~k~~ global fields k

$\exists X/k$ s.t. $X(k) = \emptyset$ & $X(\mathbb{A}_k)^{et,Br} \neq \emptyset$

& HS, CTPS, Smeets

Common

lower dim'l

$$X \longrightarrow Z \xrightarrow{g} \mathbb{P}^1$$

s.t. $g(Z(k))$ is finite.