Rational pts on surfaces

<u>Q</u> Given a global field k, & a k-variety X, how do we determine if $X(k) \neq \emptyset$?

<u>Example</u> (Lind, Reichardt)

$$C: 2Y^2 = X^4 - 17Z^4 \longleftrightarrow \begin{array}{l} 2y^2 = w^2 - 17z^2 \\ zw = x^2 \end{array} \quad /\mathbb{Q}$$

$$Z \longleftrightarrow z$$
$$X \longleftrightarrow x$$
$$Y \longleftrightarrow yz$$

<u>Claim</u>

$C(\mathbb{R}) \neq \emptyset$ & $C(\mathbb{Q}_p) \neq \emptyset$ $\forall p$

clear

If $p \neq 2, 17$ then $C/\mathbb{F}_p$ is smooth

$C(\mathbb{F}_p) \neq \emptyset \Rightarrow C(\mathbb{Q}_p) \neq \emptyset$

Weil conj for curves :

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If $C/\mathbb{Q}$ is a smooth curve of genus g & $C/\mathbb{F}_p$ is smooth $\#C(\mathbb{F}_p) \geq p - 2g\sqrt{p} + 1$

For this ex.

$$\#C(\mathbb{F}_p) \geq \cancel{p - \sqrt{}}\,(\sqrt{p}-1)^2 > 0$$

Check by hand $C(\mathbb{Q}_2) \neq \emptyset$ & $C(\mathbb{Q}_{17}) \neq \emptyset$

<u>Claim 2</u> $C(\mathbb{Q}) = \emptyset$

Assume $[x_0, y_0, z_0] \in C(\mathbb{Q})$ WMA $x_0, y_0, z_0 \in \mathbb{Z}$ pairwise rel. prime

Let $p \mid y_0$ $p > 2$ <u>then</u> $\left(\frac{17}{p}\right) = 1$

Since $17 \equiv 1 (4)$ $\boxed{\text{by QR}}$ $\left(\frac{p}{17}\right) = 1$

Since $17 \equiv 1 (8)$ $\left(\frac{2}{17}\right) = \left(\frac{-1}{17}\right) = 1 \Rightarrow y_0 \equiv y_0'^2 \mod 17$

$\mathbb{F}_{17}^{\times 4} = \{1, -1, 4, -4\}$ $\qquad 2y_0'^4 \equiv x_0^4 \mod 17$

Brauer group of a field

k = field

$$\mathrm{Br}\, k = \left(\frac{\{\text{central simple algs}\}}{A \sim B \text{ if } M_n(A) \simeq M_m(B)}, \otimes \right) \simeq H^2(G_k, k^{s\times})$$

Examples

If k is finite, or if $k = \bar{k}$ then $\mathrm{Br}\, k = 0$

$$\mathrm{Br}\, \mathbb{R} \ni \mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij \qquad \mathrm{Br}\, \mathbb{R} = \langle \mathbb{H} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

$$i^2 = -1 = j^2 \quad ji = -ij$$

For any $a, b \in k^\times$, $\mathrm{char}(k) \neq 2$

$$(a,b)_{-1} = k \oplus k\cdot i \oplus k \cdot j \oplus k \cdot ij \quad \text{is csa}/k$$

$$i^2 = a \quad j^2 = b \quad ij = -ji$$

$$(a,b)_{-1} \in (\mathrm{Br}\, k)[2] \quad \& \quad (a,b)_{-1} = id \iff \exists \text{ nontriv sol'n } ax^2 + by^2 = z^2 \;/k$$

$$(a,b)_{-1} = (\sqrt{a} \to -\sqrt{a}, b)$$

$$b \in N(k\sqrt{a})^\times \iff ax^2 + by^2 = z^2$$

If k is an arb. field

Let L/k be a cyclic ext'n of deg n

$\sigma \in \mathrm{Gal}(L/k)$ generator

$b \in k^\times$

$$(\sigma, b) := L\langle y \mid y\alpha = \sigma(\alpha) y \;\; \forall \alpha \in L\rangle \big/ \langle y^n - b\rangle$$

<u>Claim</u> → (σ, b) is a csa/k

→ $(\sigma, b) \in \ker(\mathrm{Br}\, k \longrightarrow \mathrm{Br}\, L) =: \mathrm{Br}(L/k)$

$(\sigma, b) \in (\mathrm{Br}\, k)[n]$

<u>Thm</u> L/k cyclic $\sigma \in \mathrm{Gal}(L/k)$ gen.

Then

$$k^\times / N_{L/k}(L^\times) \xrightarrow{\sim} \mathrm{Br}\, L/k$$

$$b \longmapsto (\sigma, b)$$

<u>Cor</u> $(\sigma, b) = \mathrm{id}$

$\iff b \in N_{L/k}(L^\times)$

Thm (Merkurjev–Suslin)

$k \supseteq \mu_n$ $\quad$ char$(k) \nmid n$

then $\quad (\mathrm{Br}\, k)[n] = \langle (\sigma, b) : \begin{array}{l} \sigma \in \mathrm{Gal}(L/k) \leftarrow \text{cyclic of deg } n \\ b \in k^\times \end{array} \rangle$

Two key examples from class field thy

k nonarch. local field

$$\mathrm{inv} : \mathrm{Br}\, k \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

If L/k cyclic of deg n & unram & $\sigma \in \mathrm{Gal}(L/k)$ $\bar{\sigma} = \mathrm{Frob}$.

$$\mathrm{inv}((\sigma, b)) = \frac{v(b)}{[L:k]} \in \mathbb{Q}/\mathbb{Z}$$

k global field

$$0 \to \operatorname{Br} k \longrightarrow \bigoplus_{v \in \Omega_k} \operatorname{Br} k_v \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact.

$$\left(\begin{array}{l} \operatorname{Br} \mathbb{R} \xrightarrow{\operatorname{inv}} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z} \\ \operatorname{Br} \mathbb{C} \longrightarrow \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \end{array}\right.$$

Encodes quad. reciprocity

$p, q \in \mathbb{Q}^\times$ odd primes

$(p,q)_{-1} \in \operatorname{Br} \mathbb{Q} \implies \sum \operatorname{inv}_v (p,q)_{-1} = 0$

$\operatorname{inv}_v (p,q)_{-1} = 0$ if $v \nmid p, q, 2$

$$\left.\begin{array}{l} (-1, p) \\ (2, p) \end{array}\right\} p \text{ odd}$$

Let K be a field, let v be discrete val on K & $n \in \mathbb{Z}^+$ inv. in $\mathbb{F}_v$

<u>Def</u> There is a residue map at v

$$\partial_v : (\mathrm{Br}\, K)[n] \longrightarrow H^1(\mathbb{F}_v, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})$$

s.t. $\ker \partial_v = (\mathrm{Br}\, R_v)[n]$

If L/K is unram at v & $[L:K] = p$ prime

$\underline{\partial_v((L/K, b)) = 0} \iff$ v splits comp. in L or $v(b) \equiv 0 \bmod p$

ind of choice of gen $\sigma \in \mathrm{Gal}(L/K)$

Go back to $C: 2Y^2 = X^4 - 17Z^4$

$$\updownarrow$$

$$2y^2 = w^2 - 17z^2$$
$$zw = x^2$$

$$(17, w/y)_{-1} \in \mathrm{Br}(k(\underline{C}))$$

Can check: $\forall v$ disc. val. on $k(C)$

$$\partial_v((17, w/y)) = 0$$

Assume $[x_0 : y_0 : z_0 : w_0] \in C(\mathbb{Q})$

Can check all coors nonzero

$$(17, w_0/y_0) = (17, z_0/y_0)$$

$$\frac{w_0 z_0}{w_0^2} = \left(\frac{x_0}{y_0}\right)^2$$

$$\sum \mathrm{inv}_v((17, w_0/y_0)_v) = 1/2$$