IV Brauer groups of K3 surfaces II

Last time:

X K3/$\mathbb{C}$ $\quad \operatorname{Br} X := H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ (GAGA)

$T_X := (NS\,X)^\perp \subseteq H^2(X,\mathbb{Z}) \simeq \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

Saw:

$$\left\{\begin{array}{c}\text{cyclic subgroups of}\\ \operatorname{Br} X \text{ of order } n\end{array}\right\} \xleftrightarrow{1-1} \{\text{surjections } T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}\}$$

$$\xleftrightarrow{1-1} \left\{\begin{array}{c}\text{sublattices } \Gamma \subseteq T_X\\ \text{index } n + \text{cyclic}\\ \text{quotient}\end{array}\right\}$$

Special case: $n=2$ $\quad NS_X \simeq \mathbb{Z}h \quad h^2=2$

$$\Rightarrow T_X \simeq \langle v\rangle \oplus \underbrace{U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}}_{\Lambda'}$$

$v = e-f$

$h = e+f$

$$U = \begin{array}{c|cc} & e & f \\ \hline e & 0 & 1 \\ f & 1 & 0 \end{array}$$

Let $\Gamma_\alpha := \ker(\alpha : T_X \twoheadrightarrow \mathbb{Z}/2\mathbb{Z})$

$\exists$ 3 possibilities for Γ_α up to isometry.

Example: $\Gamma_\alpha = \langle 2v \rangle \oplus \Lambda'$ "even case"

<u>Exercise:</u> Γ_α can be <u>primitively</u> re-embedded into Λ_{K3}. $i: \Gamma_\alpha \longrightarrow \Lambda_{K3}$.

On the other hand: $H^{2,0}(X) \simeq \mathbb{C}\omega_X$

$\omega_X \in T_X \otimes \mathbb{C}$

$\Rightarrow \omega_X \in \Gamma_\alpha \otimes \mathbb{C}$

$\Rightarrow i_{\mathbb{C}}(\mathbb{C}\omega_X) \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$

lies in Ω

surjectivity of period map:

$\exists$ K3 Y with $\mathbb{C}\omega_Y = i_{\mathbb{C}}(\mathbb{C}\omega_X)$
and $T_Y \simeq i(\Gamma_\alpha)$.

$$(X, \alpha) \rightsquigarrow Y \text{ K3 degree } 8.$$

K3 deg 2
$NS\,X \simeq \mathbb{Z}$

even class
in $Br\,X[2]$

Can we go the other way? Mukai.

First: what about the other isomorphism
classes of Γ_α?

Theorem: X K3$/\mathbb{C}$ $\mathrm{NS}\,X \simeq \mathbb{Z}h$ $h^2 = 2$

$$\Gamma_\alpha = \ker(\alpha : T_X \twoheadrightarrow \mathbb{Z}/2\mathbb{Z})$$

1) If $\Gamma_\alpha^* / \Gamma_\alpha \simeq (\mathbb{Z}/2\mathbb{Z})^3$ then.

$$\Gamma_\alpha(-1) \simeq \langle h_1^2, h_1 h_2, h_2^2 \rangle^\perp \subseteq H^4(Y, \mathbb{Z})$$

where

$$\begin{tikzcd} & Y \arrow[dl, "\pi_1"'] \arrow[d, "2:1"] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 & \mathbb{P}^2 \times \mathbb{P}^2 \arrow[l] \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

$h_i = \pi_i^* \mathcal{O}(1)$.

branched along type $(2,2)$ divisor

2) If $\Gamma_\alpha \in$ "even class"

then $\Gamma_\alpha \simeq T_Y$ $\quad Y$ K3 of degree 8.

3) If $\Gamma_\alpha \in$ "odd class" then

$$\Gamma_\alpha(-1) \simeq \langle h^2, P\rangle^{\perp} \subseteq H^4(Y, \mathbb{Z})$$

where $Y \subseteq \mathbb{P}^5$ is a cubic 4-fold
that contains a plane P
h = hyperplane class.

So $(X,\alpha) \rightsquigarrow Y$ always!

Proof: Nikulin.

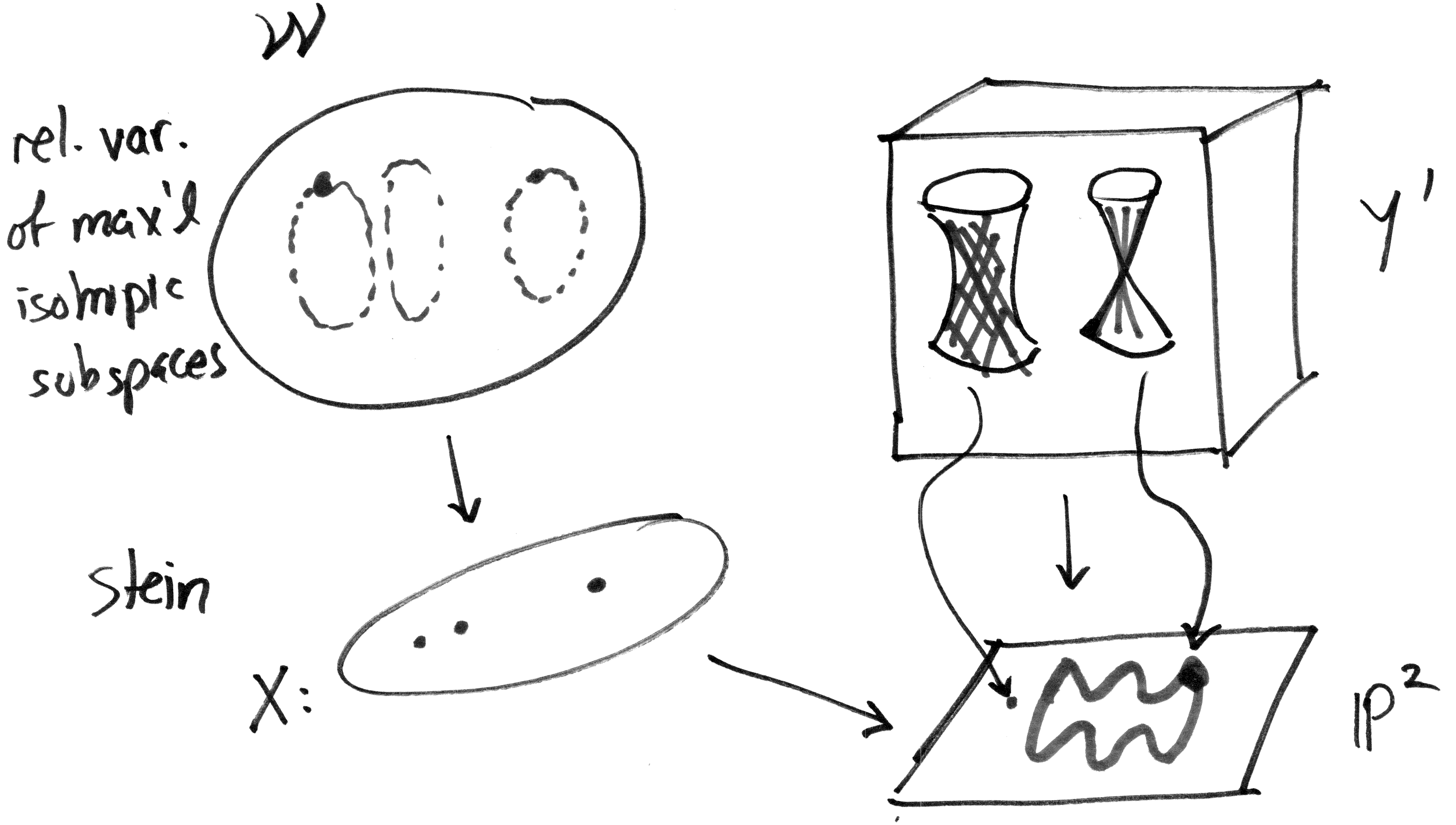
Where is the geometry?

Idea: use auxiliary variety to construct a bundle of quadrics $Y' \longrightarrow \mathbb{P}^2$

eg (Y, P) cubic 4-fold w/ plane

$$\begin{tikzcd}
Y' := Bl_P Y \arrow[d] \arrow[dr] & \\
Y \arrow[r, dashed, "\text{project away from } P"'] & \mathbb{P}^2
\end{tikzcd}$$

$Y \subset \mathbb{P}^5$



$\mathcal{W} \longrightarrow X$ is α (!!!)

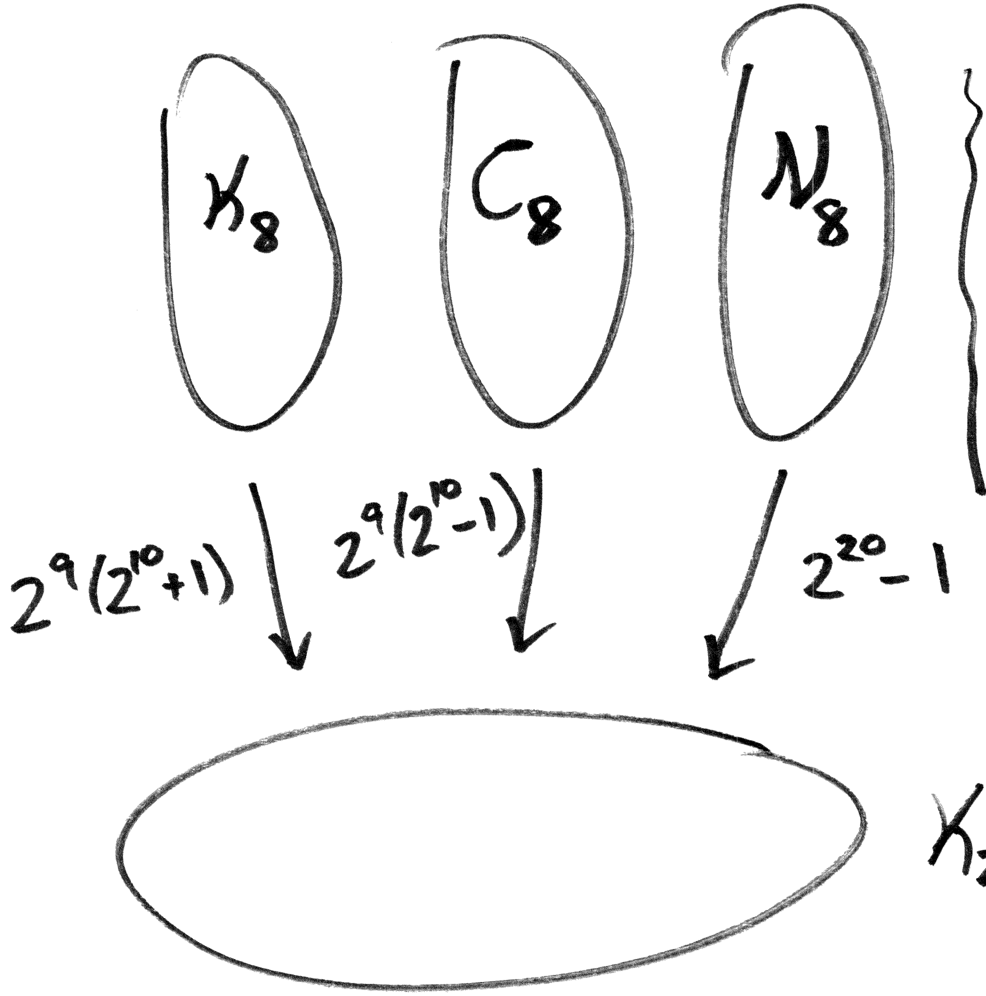
Theorem [Hassett–VA '13]

$\exists$ K3/$\mathbb{Q}$ of degree 2 with $NS\,X \simeq \mathbb{Z}$

with $\alpha \in Br\,X[2]$ transcendental such that

$X(\mathbb{A}) \neq \emptyset$ but $X(\mathbb{A})^{\alpha} = \emptyset$.

Morally:



$\mathcal{K}_2(\langle\alpha\rangle)$

$\langle\alpha\rangle \subseteq \mathrm{Br}\, X[2]$

$\mathcal{K}_2$ moduli of K3s of degree 2.

What about other elts in Br X?

Ieronymou / Skorobogatov ~~[illegible]~~ (diagonal quartics)

~~Zh~~ Zarhin

Thm (Newton '15) $E/\mathbb{Q}$ ell. curve w/ full CM

$X := \mathrm{Kum}(E \times E)$. Suppose that $(\mathrm{Br}\,X / \mathrm{Br}_1 X)_{\mathrm{odd}} \neq 0$.

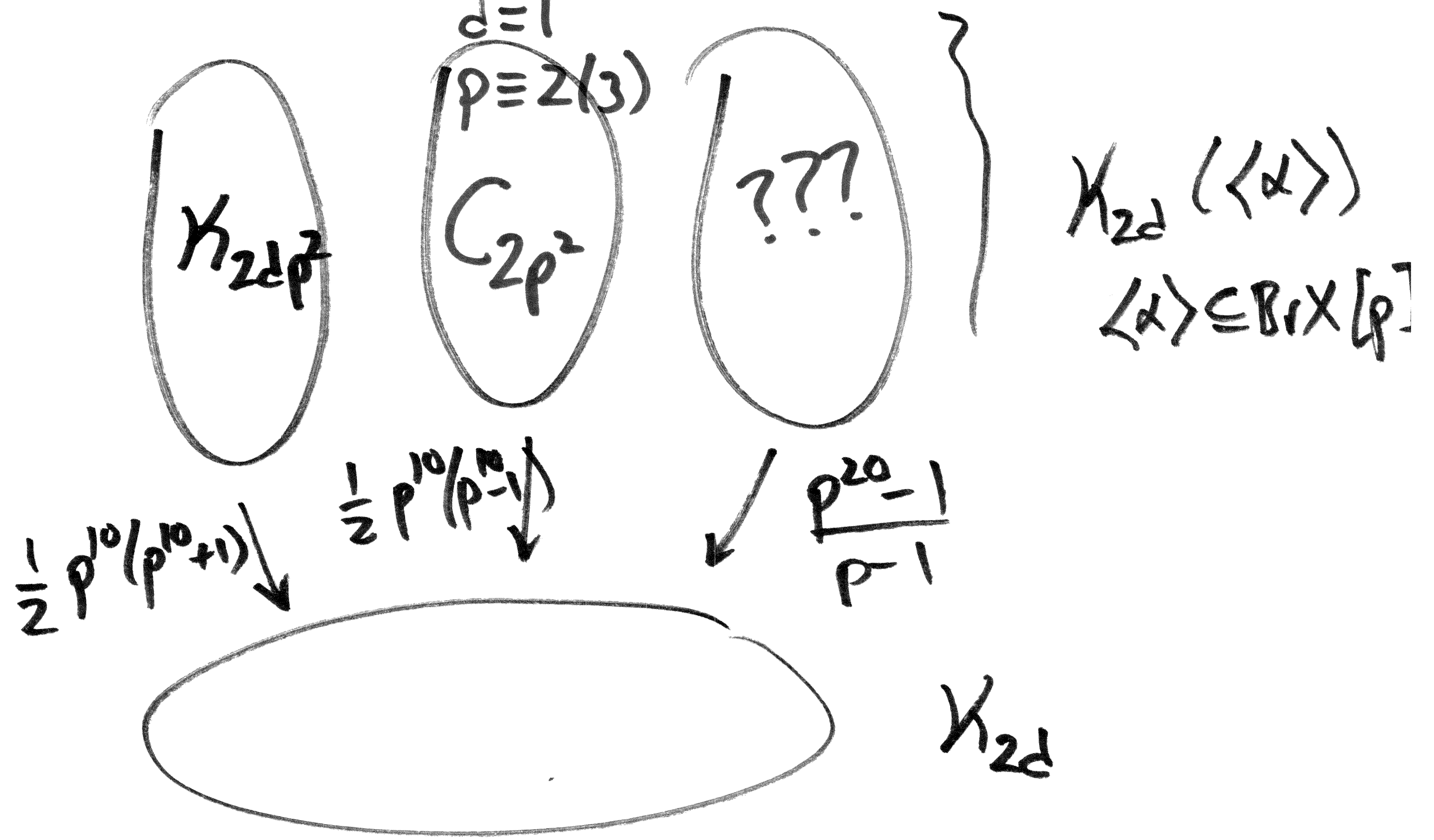
Then:

$$\mathrm{Br}_1 X = \mathrm{Br}\,\mathbb{Q} \qquad \mathrm{Br}\,X / \mathrm{Br}\,\mathbb{Q} \simeq \mathbb{Z}/3\mathbb{Z}$$

Moreover $\quad X(\mathbb{A})^{\mathrm{Br}} \subsetneq X(\mathbb{A})$

Theorem: X K3/$\mathbb{C}$ $\quad \mathrm{NS}\,X \simeq \mathbb{Z}h \quad h^2 = 2d$

ptd $\quad$ (McKinnie, Sawon, Tanimoto, VA '14)



<u>Challenge:</u> $d=1$ $p=3$

$\mathcal{C}_{18}$ = cubic 4-folds w/ dP6

Produce $X \in \mathcal{K}_2$ with $0 \neq \alpha \in \mathrm{Br}\, X[3]$.