X K3 surface / $\mathbb{Q}$ $p \neq 2$ prime good reduction.

Elsenhans-Jahnel

$$NS(\bar{X}) \hookrightarrow NS(\bar{X}_p)$$

has torsion free cokernel.

Great!

- one prime can suffice if some mod p generator does not lift.

III Brauer groups of K3 surfaces I

X nice variety / # field k

Filtration of $\operatorname{Br} X := H^2_{\text{ét}}(X, \mathbb{G}_m)$

$$\begin{array}{ccccc} \operatorname{Br}_0 X & \subseteq & \operatorname{Br}_1 X & \subseteq & \operatorname{Br} X \\ \| & & \| & & \\ \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) & & \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X}) & & \end{array}$$

"constant" algebraic classes

(injective if $X(\mathbb{A}) \neq \emptyset$)

$\operatorname{Br} X \setminus \operatorname{Br}_1 X$ = transcendental elements

For algebraic classes

$$\mathrm{Br}_1 X / \mathrm{Br}_0 X \xrightarrow{\sim} H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}\,\bar{X})$$

<u>Fact</u>: If X is [a curve] or [a surface with $\kappa(X) = -\infty$]
then $\mathrm{Br}\,\bar{X} = 0$.

Viray's lecture: X K3 surface / $\mathbb{C}$

$$\mathrm{Br}\,\bar{X} \simeq (\mathbb{Q}/\mathbb{Z})^{22-\rho(\bar{X})}$$

<u>However</u>:

Thm (Skorobogatov, Zarhin '08) X K3 / # field
$\mathrm{Br}X / \mathrm{Br}_0 X$ is finite!

Q. How do we write down trans elts on a K3?

$$\mathrm{Br}\,X \hookrightarrow \mathrm{Br}\,k(X)$$

One approach: lattices + Hodge theory.

Lattices

L = free ab. gp finite rank

$\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ sym. non-degenerate bilinear form. | even

extend to $L \otimes \mathbb{Q}$ $\mathbb{Q}$-linearity

Dual lattice: $L^* := \mathrm{Hom}(L, \mathbb{Z}) \simeq \{x \in L \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z}\ \forall\, y \in L\}$

Discriminant group: L^*/L

Discrimant form: $q_L : L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$

$$x + L \longmapsto \langle x, x \rangle \bmod 2\mathbb{Z}$$

$\ell(L) = $ min'l # generators of L^*/L as ab gp.

Theorem (Nikulin '79): L even + indefinite and $\mathrm{rk}\, L \geq \ell(L) + 2$ then L is determined up to isometry by its rank, signature + discriminant form.

$L \hookrightarrow M$ is <u>primitive</u> if M/L is torsion free

eg 1) $L \hookrightarrow M$ any embedding $\Rightarrow L^\perp$ is primitive.

2) $(L^\perp)^\perp = L \Leftrightarrow L \hookrightarrow M$ is primitive.

Transcendental Brauer groups.

$X/\mathbb{C}$ K3 surface $\qquad \operatorname{Br} X := H^2(X, \mathcal{O}_X^*)_{\text{tors}}$.

Exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$

$$
\begin{tikzcd}
H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d] & \overset{\mathrm{NS}X}{\overset{\rotatebox{90}{$\cong$}}{H^1(X,\mathcal{O}_X^\times)}} \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathcal{O}_X) \arrow[dll, bend left] \\
0 & H^2(X,\mathcal{O}_X^*) \arrow[r] & H^3(X,\mathbb{Z}) \arrow[d] & \\
 & & 0 &
\end{tikzcd}
$$

$$\Rightarrow 0 \to H^2(X,\mathbb{Z})/c_1(\mathrm{NS}X) \longrightarrow H^2(X,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X^\times) \to 0$$

Apply $\mathrm{Tor}_{\cdot}^{\mathbb{Z}}(\,\cdot\,, \mathbb{Q}/\mathbb{Z})$:

$$\mathrm{Br}\,X \xrightarrow{\sim} H^2(X,\mathbb{Z})/c_1(NS(X)) \otimes \mathbb{Q}/\mathbb{Z}$$

Understand RHS:

$$T_X := (NS\,X)^{\perp} \subseteq H^2(X,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

Proposition:

$$\phi : H^2(X,\mathbb{Z})/c_1(NS(X)) \longrightarrow T_X^*$$
$$v + NS\,X \longmapsto [t \mapsto \langle v,t\rangle]$$

is an isometry.

Proof uses
- $NS\,X$ + T_X are primitive in $H^2(X,\mathbb{Z})$.
  $\Rightarrow T_X^\perp = NS(X) \rightsquigarrow$ injectivity.
- surjectivity: $H^2(X,\mathbb{Z})$ is unimodular.

<u>Conclusion:</u> $\mathrm{Br}\,X \simeq T_X^{\times} \otimes \mathbb{Q}/\mathbb{Z}$

$\simeq \mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$.

$\therefore$

$$\left\{\begin{array}{l}\text{cyclic subgroups of} \\ \mathrm{Br}\,X \text{ of order } n\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\text{surjections } T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}\right\}$$

$$\overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{index } n \text{ sublattices} \\ \Gamma \subseteq T_X \text{ with} \\ \text{cyclic quotient}\end{array}\right\}$$

Simplest example:

X complex K3 w/ $NS\,X \simeq \mathbb{Z}h \quad h^2 = 2$

(degree 2)

Look for lattices

$$\Gamma \underset{2}{\subsetneq} T_X \longleftrightarrow 0 \neq \alpha \in \mathrm{Br}\,X[2]$$

$$NS(X) = \langle h \rangle \hookrightarrow \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

$$h \longmapsto e+f$$

$$T_X = \langle v \rangle \oplus \overbrace{U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}}^{\Lambda' \text{ unimodular}} \qquad U \quad \begin{array}{c|cc} & e & f \\ \hline e & 0 & 1 \\ f & 1 & 0 \end{array}$$

where $v = e - f$

Surjections: $\alpha : T_X \longrightarrow \mathbb{Z}/2\mathbb{Z}$

$$nv + \lambda' \longmapsto n a_\alpha + \langle \lambda', \lambda_\alpha \rangle \bmod 2\mathbb{Z}$$

$\lambda' \in \Lambda'$

$a_\alpha \in \{0,1\}$

$\lambda_\alpha \in \Lambda'$ well-defined up elt in $2\Lambda'$.

Theorem (van Geemen):

$\Gamma_\alpha := \ker \alpha$

1) $a_\alpha = 0 \Rightarrow \Gamma_\alpha^\times / \Gamma_\alpha \simeq (\mathbb{Z}/2\mathbb{Z})^3$

All such lattices are isomorphic. $2^{20}-1$ lattices.

2) $a_\alpha = 1 \Rightarrow \Gamma_\alpha^\times / \Gamma_\alpha \simeq \mathbb{Z}/8\mathbb{Z}$

Two isomorphism classes:

$2^9(2^{10}+1)$

i) even class: $\frac{1}{2}\langle \lambda_\alpha, \lambda_\alpha \rangle \equiv 0 \bmod 2$

ii) odd class: $\frac{1}{2}\langle \lambda_\alpha, \lambda_\alpha \rangle \equiv 1 \bmod 2$

$2^9(2^{10}-1)$

Example: $\Gamma = \langle 2v \rangle \oplus \Lambda' \subseteq \langle v \rangle \oplus \Lambda' = T_X$

$= \ker(\alpha : T_X \longrightarrow \mathbb{Z}/2\mathbb{Z})$

$nv + \lambda' \longmapsto n \bmod 2$

$(X, \alpha) \rightsquigarrow Y$ degree 8

K3 degree

even class

WHOA!