II Picard numbers of K3 surfaces

X = algebraic K3 surface / k # field
(geom. int.) $\omega_X \simeq \mathcal{O}_X$; $H^1(X, \mathcal{O}_X) = 0$.

$\overline{X} := X \times_k \overline{k}$

$\mathrm{Gal}(\overline{k}/k)$-module structure of $\mathrm{Pic}\,\overline{X}$
encodes arithmetic on X, eg
$\mathrm{Br}_1 X / \mathrm{Br}_0 X$ if $X(\mathbb{A}_k) \neq \emptyset$.

Question: What is $\mathrm{rk}\,\mathrm{Pic}\,\overline{X} =: \rho(\overline{X})$?

Recall $\mathrm{Pic}\,\bar{X} \simeq \mathrm{NS}\,\bar{X} \simeq \mathrm{Num}\,\bar{X}$ and

$$1 \leq \rho(\bar{X}) \leq 20$$

§1: reduction ~~map~~ mod p / specialization.

<u>Good reduction</u>:

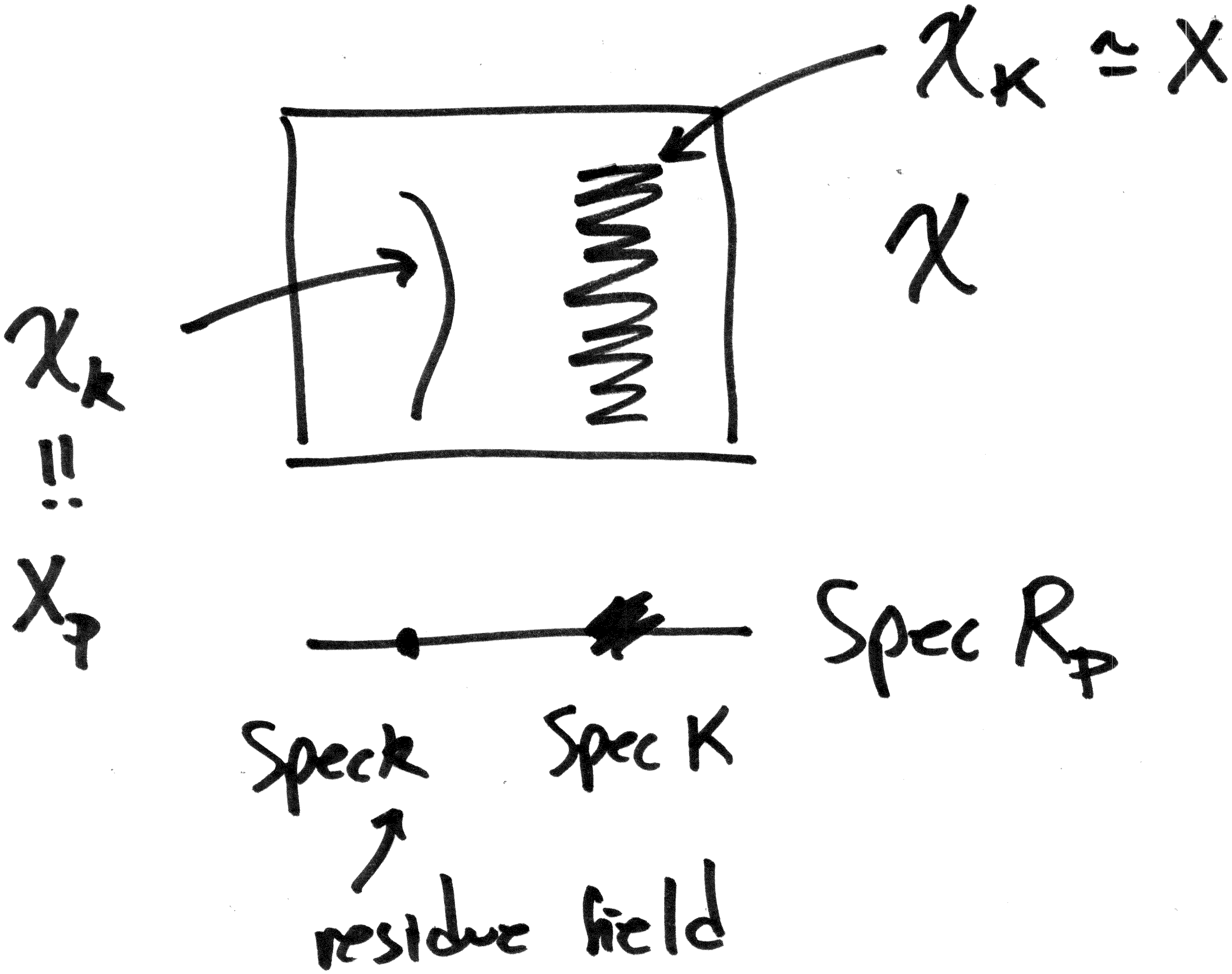
$\mathrm{Frac}\,R = K \supseteq R$ Dedekind domain $\quad \mathfrak{p}$ nonzero prime

X = smooth proper K-variety.

good reduction at $\mathfrak{p}$:

$\exists$ smooth proper $\mathcal{X} \longrightarrow \mathrm{Spec}\, R_{\mathfrak{p}}$

s.t. $\mathcal{X}_K \simeq X$ as K-schemes



$\mathcal{X}_K \simeq X$
$\mathcal{X}$
$\mathcal{X}_k$
||
X_p
Spec R_p
Spec k
Spec K
residue field

Specialization

integral closure

$$
\begin{tikzcd}
\bar k := \bar R/\mathfrak{p} \arrow[d, no head, bend right] & \bar R \arrow[l] \arrow[d, no head] & \bar K \arrow[d, no head] \\
k & \underset{\text{DVR}}{R} \arrow[l] \arrow[r, phantom, "\subseteq"] & K
\end{tikzcd}
$$

$p = \operatorname{char} k$.

$\mathcal{X} \longrightarrow \operatorname{Spec} R$ smooth (rel. dim 2) proper.

$\exists \quad \operatorname{Pic} \mathcal{X}_{\bar K} \longrightarrow \operatorname{Pic} \mathcal{X}_{\bar k}$

$\rightsquigarrow sp_{\bar K, \bar k} : NS(\mathcal{X}_{\bar K}) \longrightarrow NS(\mathcal{X}_{\bar k})$

injective up to p-torsion.

Conclusion: (K number field)

If $X_{\bar{K}}, X_{\bar{k}}$ are K3 surfaces then

$$\rho(X_{\bar{K}}) \leq \rho(X_{\bar{k}})$$

§2 cycle class map.

$X/\mathbb{F}_q$ "nice" $q = p^r$ $\ell \neq p$ prime

$\sigma \in \mathrm{Frob} \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) =: \Gamma$

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 0 \quad \text{Kummer}$$

exact on $\bar{X}_{ét}$

$$\rightsquigarrow \quad \delta_n : H^1_{ét}(\bar{X}, \mathbb{G}_m) \longrightarrow H^2_{ét}(\bar{X}, \mu_{\ell^n})$$

$$\begin{array}{ccc} \wr\!\wr & & \| \\ \operatorname{Pic} \bar{X} & & \mathbb{Z}/\ell^n\mathbb{Z}(1) \end{array}$$

Take $\varprojlim$

$$\operatorname{Pic} \bar{X} \longrightarrow \varprojlim H^2_{ét}(\bar{X}, \mu_{\ell^n}) =: H^2_{ét}(\bar{X}, \mathbb{Z}_\ell(1))$$

tensor $\otimes \mathbb{Q}_\ell$

$$c: NS(\bar{X}) \otimes \mathbb{Q}_\ell \hookrightarrow H^2_{ét}(\bar{X}, \mathbb{Q}_\ell(1))$$

$$H^2_{ét}(\bar{X}, \mathbb{Q}_\ell(1)) \simeq H^2_{ét}(\bar{X}, \mathbb{Q}_\ell)(1)$$

$$\updownarrow$$

$$H^2 \otimes_{\mathbb{Q}_\ell} (\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n})$$

c is compatible with Γ-actions.

<u>Observation:</u>

Some power of $\sigma^* \curvearrowright NS(\bar{X})_{\mathbb{Q}_\ell}$ acts as id.

$\Rightarrow$ eigenvalues of $\sigma^* \curvearrowright NS(\bar{X})_{\mathbb{Q}_\ell}$ are roots of 1.

put together:

$\rho(\bar{X}) = \operatorname{rk} NS(\bar{X})_{\mathbb{Q}_\ell} \leqslant$ # eigenvalues of $\sigma^*(1)$ on $H^2_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)(1)$ that are roots of unity.

Take conjecture: this is an equality (OK if q odd)

$\leqslant$ # eigenvalues of $\sigma^* \curvearrowright H^2_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$ of the form $\frac{\nu}{q}$ where ν is a root of unity.

Want: charac poly of $\sigma^* \curvearrowright H^2_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$.

Can get: charac poly of $(\sigma^*)^{-1} \curvearrowright H^2_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$.

X is K3 surface / k # field $\mathfrak{p}$ prime of good reduction.

Then

$$\rho(\bar{X}) \underset{sp}{\leq} \rho(\bar{X}_{\mathfrak{p}}) \leq \text{\# of eigenvalues of } (\sigma^*)^{-1} \curvearrowright H^2_{ét}(\bar{X}, \mathbb{Q}_\ell) \text{ of the form } \gamma \cdot q \text{ where } \gamma = \text{root of } 1.$$

⚠ this number is <u>even</u>.

Let $\Psi_q(x)$ = charac poly of $(\sigma^*)^{-1} \curvearrowright H^2_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)$

Linear algebra: $\Psi_q(x)$ can be computed from traces of powers of $(\sigma^*)^{-1}$!

Good news!

Lefschetz trace formula:

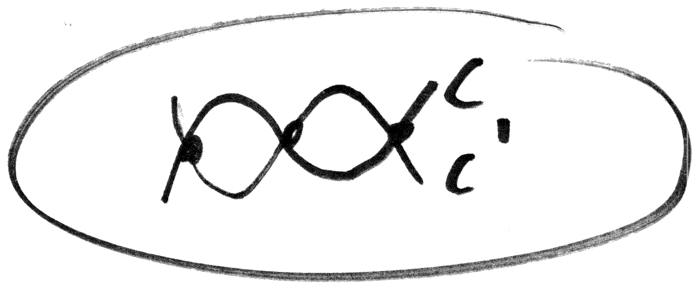
$$\mathrm{Trace}\,(\sigma^{*-1})^n = \#X_p(\mathbb{F}_{q^n}) - 1 - q^{2n}.$$

⚠ counting points is hard!

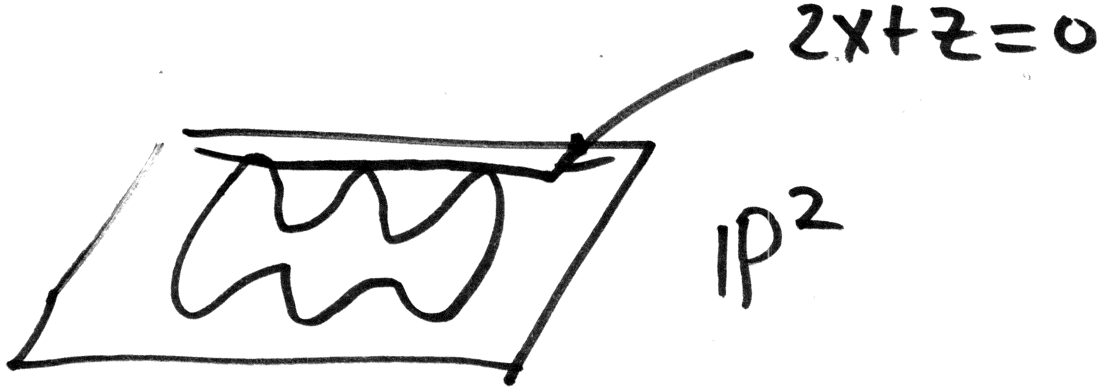
Example: $X \subseteq \mathbb{P}_{\mathbb{F}_3}(1,1,1,3)$

$$w^2 = 2y^2(x^2+2xy+2y^2)^2 + (2x+z)p_5(x,y,z)$$

Can show $\rho(\bar{X}) \leq 2$



	C	C'
C	-2	3
C'	3	-2



Conquering $\rho(\overline{X})=1$ (van Luijk)

$k=\mathbb{Q}$ for simplicity

X : K3 surface p, p' <u>two</u> primes of good reduction.

Suppose that

$$
\begin{tikzcd}
 & NS(\overline{X}_p) \simeq \mathbb{Z}^n \\
NS(\overline{X}) \arrow[ur, hook] \arrow[dr, hook] & \\
 & NS(\overline{X}_{p'}) \simeq \mathbb{Z}^n
\end{tikzcd}
$$

If $\operatorname{Disc} NS(\overline{X}_p) \neq \operatorname{Disc} NS(\overline{X}_{p'})$ is $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$

then $\rho(\overline{X}) \leq n-1$.

To compute Disc $NS(\bar{X}_p)$ either

- find explicit generator.
- use Artin-Tate formula (Kloosterman)

[Elsenhans-Jahnel: $p \neq 2$ $k = \mathbb{Q}$.

$NS(\bar{X}) \hookrightarrow NS(\bar{X}_p)$ has torsion-free cokernel.]