Arithmetic of K3 surfaces.

I. Geometry ($k = \bar{k}$)

surface X/k: smooth projective integral.

<u>Algebraic</u> K3 surface: $\omega_X \simeq \mathcal{O}_X$; $H^1(X, \mathcal{O}_X) = 0$.

<u>Examples:</u>

1) Quartic surfaces in $\mathbb{P}^3$ "degree 4"
2) (Quadric) $\cap$ (Cubic) in $\mathbb{P}^4$ "degree 6"
3) $V(Q_1, Q_2, Q_3)$ in $\mathbb{P}^5$ "degree 8"

char $k \neq 2$.

4) $X \xrightarrow{2:1} \mathbb{P}^2$ branched along smooth sextic $C \in \mathbb{P}^2$ "degree 2".

$$X : \ w^2 = f_6(x,y,z) \ \text{in} \ \mathbb{P}(1,1,1,3)$$

$$\begin{array}{c} x\ y\ z\ w \\ \downarrow \\ \mathbb{P}(1,1,1) \end{array}$$

5) A abelian surface $\quad i : A \xrightarrow{[-1]} A$ 16 fixed pts $A[2]$

$\tilde{A}$ = blow-up of A along $A[2]$

lift $\tilde{i} : \tilde{A} \to \tilde{A}$

$X := \tilde{A} / \tilde{i}$ Kummer K3 associated to A.

Intersection pairing $(\ ,\ )_X : \operatorname{Div} X \times \operatorname{Div} X \longrightarrow \mathbb{Z}$.

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r] \arrow[rr, bend right, "\simeq \text{ isom for K3s.}"'] & NS(X) \arrow[r] & \operatorname{Num}(X) \end{tikzcd}$$

torsion free.

The case $k = \mathbb{C}$.

<u>Complex</u> K3 surface : compact connected 2-dim'l complex manifold with $\omega_X = \Omega_X^2 \cong \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$.

$$\{\text{algebraic K3s}/\mathbb{C}\} \xhookrightarrow[\text{GAGA}]{} \{\text{Complex K3s}\}$$

$$X \longmapsto X^{an}.$$

$$\frac{1}{12}\Big(\underset{0}{\cancel{c_1^2}}+c_2\Big) \underset{\text{Noether}}{=} \chi(X,\mathcal{O}_X) = \underset{1}{h^0(\mathcal{O}_X)} - \underset{0}{h^1(\mathcal{O}_X)} + \underset{1}{h^2(\mathcal{O}_X)}$$

$$= 2$$

$$c_2 = \chi_{top}(X) = 24.$$

Algebraic topology:

$H^0(X,\mathbb{Z}) \simeq \mathbb{Z}$ oriented

$\updownarrow$

$H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$ connected

Exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$

$$H^0(\mathcal{O}_X) \twoheadrightarrow H^0(\mathcal{O}_X^*) \to H^1(\mathbb{Z}) \to \cancelto{0}{H^1(\mathcal{O}_X)}$$

$\Rightarrow H^1(X,\mathbb{Z}) = 0.$

Next: $0 = \mathrm{rk}\, H^1(X,\mathbb{Z}) \underset{UC}{=} \mathrm{rk}\, H_1(X,\mathbb{Z}) \overset{PD}{=} \mathrm{rk}\, H^3(X,\mathbb{Z})$

$\Rightarrow H^3(X,\mathbb{Z})$ is torsion!

$$H^3(X,\mathbb{Z})^\vee_{\mathrm{tors}} \overset{PD}{\simeq} H_1(X,\mathbb{Z})_{\mathrm{tors}} \overset{UC}{\simeq} H^2(X,\mathbb{Z})_{\mathrm{tors}}.$$

<u>Proposition</u>: $H_1(X,\mathbb{Z})_{\mathrm{tors}} = 0$.

Hence $H^3(X,\mathbb{Z}) = 0$ and $H^2(X,\mathbb{Z})$ is free abelian! Rank $H^2(X,\mathbb{Z}) = 24-1-1 = 22$.

Lattice structure of $H^2(X,\mathbb{Z})$

cup product $B : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$.

perfect bilinear <u>even</u> pairing!

$$B(x,x) \in 2\mathbb{Z}.$$

$$\rightsquigarrow \quad q : H^2(X,\mathbb{Z}) \longrightarrow 2\mathbb{Z}$$
$$x \longmapsto B(x,x).$$

$\rightsquigarrow$ $q_{\mathbb{R}}$ w/ signature (b_+, b_-)

Thom-Hirzebruch index thm

$$b_+ - b_- = \tfrac{1}{3}(c_1^2 - 2c_2) = \tfrac{1}{3}(0 - 2\cdot 24) = -16.$$
$$b_+ + b_- = 22$$

$\Rightarrow (b_+, b_-) = (3, 19).$

Summary: $H^2(X,\mathbb{Z})$ is even, indefinite, unimodular (PD) of signature $(3,19)$.

$$\underset{\text{Milnor}}{\Longrightarrow} \quad H^2(X,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

where $U =$ "hyperbolic plane". $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

<u>Fundamental group</u> $\pi_1(X) = 0$.

- All K3's are diffeomorphic.
- $X_4 \subseteq \mathbb{P}^3$ $\qquad \nu : \mathbb{P}^3 \hookrightarrow \mathbb{P}^{34}$ 4-uple.

$$\pi_1(X) \cong \pi_1(\nu(\mathbb{P}^3) \cap H) \underset{LHT}{\simeq} \pi_1(\nu(\mathbb{P}^3)) = 0.$$

Differential geometry

$k=1,2$ $\qquad H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$ Hodge decomposition.

$$H^{p,q}(X) \simeq H^q(X, \Omega^p_X) \quad \text{(Dolbeault)}$$

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

Hodge diamond

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & \boxed{20.} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- $\operatorname{Pic} X \xhookrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{\iota_*} H^2(X,\mathbb{C})$

Lefschetz (1,1)-thm:

$$\operatorname{im}(\iota_* \circ c_1) = H^{1,1}(X) \cap i_* H^2(X,\mathbb{Z}).$$

Consequence

$$0 \leq \underset{\substack{\parallel \\ \rho(X)}}{\operatorname{rk} \operatorname{Pic} X} \leq 20.$$

- $H^{2,0}(X) = \mathbb{C}\omega_X$

$$(\, , \,) : H^2(X,\mathbb{C}) \times H^2(X,\mathbb{C}) \longrightarrow \mathbb{C}$$
$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta .$$

Hodge-Riemann relations:

1) $(\omega_X, \omega_X) = 0$
2) $(\omega_X, \overline{\omega_X}) > 0$
3) $H^{2,0}(X) \oplus H^{0,2}(X)$ orthogonal to $H^{1,1}(X)$.

$\Rightarrow \mathbb{C}\omega_X$ determines the Hodge decomposition on $H^2(X,\mathbb{C})$.

<u>Marking</u> $\Phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

HR ~~Riemann~~ relations $\Rightarrow$ $\Phi_{\mathbb{C}}(\mathbb{C}\omega_X)$ $\leftarrow$ period point.

lies in

$$\Omega := \left\{ x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) : \begin{array}{l} (x,x) = 0 \\ (x,\bar{x}) > 0 \end{array} \right\} \subseteq \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$$

$\uparrow$

period domain.

open subset of a quadric in $\mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$.

Weak Torelli theorem:

X, X' complex K3s are isom $\Leftrightarrow$ $\exists$ markings

$$\Phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{K3} \xleftarrow{\sim} H^2(X',\mathbb{Z}) : \Phi'$$

such that $\Phi_{\mathbb{C}}(\mathbb{C}\omega_X) = \Phi'_{\mathbb{C}}(\mathbb{C}\omega_{X'})$.

[Surjectivity of period map.

$\underline{\omega \in \Omega}$ $\rightsquigarrow$ 1-dim'l space $H^{2,0} \subseteq \Lambda_{K3} \otimes \mathbb{C}$

$\rightsquigarrow \Lambda_{K3} \otimes \mathbb{C} \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

$\exists$ complex K3 surface + marking $\Phi : H^2(X,\mathbb{Z}) \to \Lambda_{K3}$

s.t. $\Phi_{\mathbb{C}}$ preserves Hodge decompositions.]