

— ARIZONA WINTER SCHOOL COURSE OUTLINE —

LINKS BETWEEN “ARITHMETIC” AND “GEOMETRIC”  
ALGEBRAIC GEOMETRY: METAPHOR AND MOTIVATION

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Course outline

The interplay among “arithmetic”, “topology”, and “geometry” has been a central theme in algebraic geometry since long before the Weil conjectures. This course is intended to give a taste of a number of related questions where ideas in one area directly imply ideas in another, or (more subtly) suggest through metaphor statements one should hope/believe/expect/prove to be true.

The **first lecture** will discuss *symmetric powers*. Symmetric powers in topology (a version of a *configuration space*) have a great deal of structure. A simple version of this is Macdonald’s formula for the Euler characteristic (with compact supports) of the symmetric power in terms of the Euler characteristic of the original space. A fancier version is the Dold-Thom theorem, which says that the  $n$ th symmetric power, as  $n \rightarrow \infty$ , “stabilizes” to something easily describable in terms of the original space.

(This is a first example of how “stabilization” as some parameter gets large can lead to more structure being visible. This will be a theme of this course. It has also been a central player in a number of Arizona Winter School courses in recent years, by Poonen, Ellenberg, and Matchett Wood. See for example [P, EVW, VW].)

By enriching the notion of Euler characteristic, we are led to Grothendieck’s roadmap for proving the Weil conjectures (except for the Riemann hypothesis). The topology tells us precisely what we want from the cohomology theory. But by enriching it in different ways, we get geometric information too — for example, on (mixed) Hodge structures of symmetric powers — which was proved surprisingly long after the Weil conjectures (in Jan Cheah’s 1994 Ph.D. thesis).

The **second lecture** will continue this theme. By trying to make sense of “Euler characteristic” in as much generality, we are led to the definition of the *Grothendieck ring of varieties*. Fix a field  $k$ . The Grothendieck ring  $K = K_0(\text{Var}_k)$  of varieties over  $k$  is generated as an abelian group by the classes of finite type  $k$ -schemes up to isomorphism. If  $Y$  is a closed subscheme of  $X$ , and  $U$  is its (open) complement, then we impose the condition  $[X] = [U] + [Y]$ . Defining a product by  $[X][Y] := [X \times_k Y]$  makes  $K$  into a commutative ring, with  $[\text{Spec } k]$  as unit. A number of important properties factor through this ring, including the point-counting map (if  $k$  is a finite field), and Hodge structures. This ring has some

surprisingly pathological/interesting properties (see for example the ingenious ideas of [LL]).

Define the “Lefschetz motive”  $\mathbb{L} := [\mathbb{A}^1]$ . There are many reasons to consider the localization  $K_{\mathbb{L}}$  (including motives; motivic integration; possible rationality of the motivic zeta function; the homotopy axiom in topology; some further reasons given in [VW]; ...). Point-counting and Hodge structures both extend to this localization.

The group  $K$  is filtered by the subgroups generated by varieties of dimension at most  $d$ , as  $d$  varies. This “dimensional filtration” extends to  $K_{\mathbb{L}}$ . The completion  $\widehat{K}_{\mathbb{L}}$  inherits not just a group structure, but also a ring structure. (This completion was first introduced by Kontsevich in his theory of motivic integration.)

Hodge structures extend to this completion, after suitably extending the codomain, but the point-counting map  $K_{\mathbb{L}} \rightarrow \mathbb{Q}$  is not continuous (consider the sequence  $(2q)^n \mathbb{L}^{-n}$  where  $k = \mathbb{F}_q$ ). Ekedahl provides a fix for this problem in [E], by giving a more refined yet geometrically natural filtration which allows for a map from the “Ekedahl completion” to  $\mathbb{R}$ .

By working with this “Euler characteristic”, one can show a number of results analogous to known results in arithmetic and in topology. And conversely, results one can show working with the Grothendieck ring lead to conjectural statements in both arithmetic and topology. (See [VW] for more.) In most cases, the cleanest statements are on the stabilizations.

The **third lecture** will describe geometric analogs of recent famous stabilization results in number theory. Bhargava’s celebrated results tell us “how many” degree  $d$  extensions of  $\mathbb{Q}$  there are, counted in the only reasonable way, [B1, B2]. The constructions he uses are geometric and ancient. For example, to count quartic extensions of  $\mathbb{Q}$ , the geometry/algebra is that of how we solve quartic equations in one variable by reducing to cubics.

On the geometric side, Mukai’s constructions of moduli spaces of curves of genus 6 through 9, and of moduli of K3 surfaces of low degree, largely parallel the classical constructions. As a more simple example, the moduli space of elliptic normal curves in  $\mathbb{P}^n$  (a nondegenerate genus 1 curve of degree  $n + 1$  in  $\mathbb{P}^1$ ) fits into this family as well.

But Bhargava’s results suggest more — that the space of curves of genus  $g$  that are degree  $d$  covers of  $\mathbb{P}^1$  should in some sense stabilize for  $d = 3, 4, 5$ , as  $g \rightarrow \infty$ . (These are called trigonal, tetragonal, and pentagonal curves, respectively; for  $d = 2$ , they are called hyperelliptic curves.) This is true for  $d = 3$  and  $d = 4$ , and not yet shown for  $d = 5$ . Bhargava also has an explicit conjecture for how his results extend beyond degree 5, and this has an unexpected and somewhat alarming geometric analogue.

The geometric results translate *back* into arithmetic geometry to suggest possible new results, by point-counting. We should be able to “count” the number of genus  $g$  hyperelliptic, trigonal, and tetragonal curves (or more precisely, understand the behavior as  $g \rightarrow \infty$ ). One might even hope to “count” genus  $g$  curves over  $\mathbb{Q}$  for small genus.

The **fourth lecture** will take as its starting point the discussion of the geometry of the space of elliptic normal curves in  $\mathbb{P}^n$  in the third lecture. This is essentially the space of genus 1 curves with a degree  $n + 1$  line bundle. The beautiful arithmetic of quadratic, cubic, and quadratic fields (and even quintic fields) suggests that there should be beautiful structure involving this moduli space, and indeed this is the case, by considering genus 1 fibrations over  $\mathbb{P}^1$  (along with a line bundle of relative degree  $d$ ). As a key base case, I will discuss the consequences for various spaces of 12 points on  $\mathbb{P}^1$ , and then discuss generalizations (due to A. Deopurkar and A. Patel). This lecture will make contact with the other courses in this Winter School, as these fibrations are conveniently understood using Brauer groups.

## Sample problems

A number of problems of different flavors and different levels of difficulty naturally come out of these ideas. Here is a representative sample.

1. How many hyperelliptic/trigonal/tetragonal genus  $g$  curves are there over  $\mathbb{F}_q$ ? Over  $\mathbb{Q}$  (counted by discriminant)? (I would expect a good answer only as  $g \rightarrow \infty$ .)

2. Even though completing the Grothendieck ring (with  $\mathbb{L}$  inverted) using the dimensional filtration seems to destroy point-counting information, Ekedahl [E] suggests that in good circumstances it should be possible to still count points. Can this be made precise? Can we use this to transport geometric results directly into arithmetic, e.g., to give a new proof of Poonen’s “Bertini theorem for finite fields” [P], and also generalizations?

3. Can one prove arithmetic results analogous to the geometric ones directly, inspired by the geometric arguments? As one example, can one generalize Poonen’s theorem in the way suggested by the geometric generalization?

4. This general philosophy suggests experimental questions as well. In a beautiful paper [EM], Elkies and McMullen explain the following fact. If you look at the “fractional” parts of  $\sqrt{1}, \sqrt{2}, \dots, \sqrt{n}$  as  $n$  gets large, they get equidistributed in  $[0, 1)$  — they look like they are randomly chosen points. But if you look at the difference between adjacent points in this set (in  $(0, 1]$ ), they no longer look random! The proof of Elkies and McMullen is very special to this situation, but one can wonder if this is the first case of some interesting phenomena. For example, we are looking at points on  $x = y^2$  for integral values of  $x$ ; what makes this curve special? Is it that it is rational? Is there something special for cube roots? (There are a number of questions in this vein.)

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