Reminder: The Grothendieck Ring R

Additive generators: $[X] \quad$ varieties/iso

Additive relations $[X] = [U] + [Z]$

Multiplication $[X \cdot Y] = [X] \cdot [Y]$

The motivic zeta function

$$Z_X(t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X]\, t^n \in R[[t]]$$

$$Z : R \longrightarrow 1 + tR[[t]] \subset R[[t]]^{\times}$$

$$+ \rightsquigarrow \times$$

example: $k = \mathbb{F}_q$

$\#: \quad Z_X(t) \xrightarrow{\text{Ex.}} \zeta_X(t) \in \mathbb{Z}[[t]]$ ← Weil zeta function.

<u>Question</u> 2000 Kapranov

$Z_X(t)$ rational?

evidence: curves.

✓ $\mathbb{C}$ Hodge structures.
TRUE!

Cheah '94

$\mathbb{F}_q$ pt counting ✓

$f \in \mathbb{C}[[t]]$

$= \dfrac{g}{h}$ $g, h \in \mathbb{C}[t]$. $\qquad fh = g$.

<u>Theorem</u> (Larsen-Lunts)

No!

$k = \mathbb{F}_q$

$$R \qquad \boxed{R_L} \qquad \boxed{\mathbb{Z}.}$$

$$Z_X(u) \qquad\qquad \zeta_X$$

$k = \mathbb{C}$ (top.).

Thm $$Z_X(t) = \frac{(1-t)^{h_1}(1-t)^{h_3}\cdots}{(1-t)^{h^0}(1-t)^{h^2}\cdots}$$

Thm (Cheah)

...

<u>Thm</u> X complex variety

$$\sum_{n\geq 0} h^{p,q}\left(H_c^r(\mathrm{Sym}^n X)\right) t^n x^p y^q (-z)^r$$

$$= \prod_{p,q,r} \left(\frac{1}{1-x^p y^q z^r t}\right)^{(-1)^r h^{p,q}(H_c^r(X))}$$

__Theorem__ ...

"cohomology-like"

$$H^i : \mathrm{Var}_k \xrightarrow[\text{functor.}]{\text{contr}} (VS^{+\text{ more.}})$$

$\hookrightarrow$ field char 0.

$\oplus \quad \otimes \quad \wedge.$

$$\chi(X) := \sum (-1)^i H^i(X) \in K(VS^+)$$

__AXIOM__ : $H^i(X) = 0$ for $i \gg 0$

<u>Theorem</u>

$$\sum_{i=0}^{\infty} \chi(\mathrm{Sym}^i X)\, t^i = \frac{1}{(1-t)^{\chi(X)}} \in K(\cdot)[[t]]$$

$$(1-t)^V = 1 - (V)t + (\Lambda^2 V)t^2 - (\Lambda^3 V)t^3 \cdots \pm (\det V)t^{\dim}$$

$$(1-t)^V (1-t)^W = (1-t)^{V \oplus W}$$

Ex $\dfrac{1}{(1-t)^V} = \sum_{n=0}^{\infty} \mathrm{Sym}^n V \, t^n$

Axiom (Künneth) $H^n(X \times Y) = \bigoplus_{i=0}^{n} H^i(X) \otimes H^{n-i}(Y)$

Axiom $G \circlearrowright T$ $\phi : T \longrightarrow T/G.$

finite gp.

$\phi^* : H^i(T/G) \longrightarrow H^i(T)$

identifies $H^i(T/G)$ with $(H^i(T))^G.$

(Groth. To.)

$$\underbrace{}_{}\quad \mathrm{Sym}^n X = X^n / S_n.$$

$$H^*(\mathrm{Sym}^n X) = \left(H^*(X^n)\right)^{S_n}$$

$$= \ldots \text{ in term of } H^*(X)$$

$$\boxed{\cancel{\sum_{i=0}^{\infty} \chi}}$$

<u>Cor.</u> Hodge structures $\Longrightarrow$ Cheah.

<u>Cor</u> $k = \mathbb{F}_q$ #. $F \circlearrowright H^r_{et}(X, \mathbb{Q}_\ell)$

$$\sum_i (-1)^i \, tr\left(F|_{H^r(X, \mathbb{Q}_\ell)}\right)$$

Larsen-Lunts:

$$\mu(X) = 1 + h^0(\Omega^1)\, t + h^0(\Omega^2)\, t^2 + \dots + h^0(\Omega^{\dim X})\, t^{\dim X}$$

smooth projective

$$\in 1 + t\mathbb{Z}[[t]]$$

$$\mu(\mathbb{L}) = 0$$

$$= \mu(\mathbb{P}^1) - \mu(pt)$$

Motivation.

$$\zeta_X(t) = \frac{P_1(t)\ \cdots\ P_{2d-1}(t)}{P_0(t)\ \cdots\ P_{2d}(t)} = \sum a_i t^i$$

$$P_{2d}(t) = (1 - q^d t)$$

$a_i \sim (q^d)^i$ ~ Hasse-Weil bounds

<u>Conj</u> (Motivic Stabilization of Sym Powers). $\boxed{\text{MSSP}}$

$$\frac{\mathrm{Sym}^n X}{\mathbb{L}^{n \dim X}} \longrightarrow \mathbb{E} \in \hat{R}_{\mathbb{L}}$$

converges.

<u>Evidence</u>. pt. counting ✓

[Stably rational variety] Hodge structures ✓

curves ✓

Topology.

$p \in X$ $\mathrm{Sym}^n X \longrightarrow \mathrm{Sym}^{n+1} X$

effective Arnav Tripathy.

$$\mathrm{Sym}^{\infty} X = \boxed{\prod K(\tilde{H}_i(X), n)}.$$

<u>Thm</u> (Tripathy): "Sym" "étale realization." commute.

X & $X \times \mathbb{A}^m$ "homotopic"

$\mathrm{Sym}^n X$ $\mathrm{Sym}^n (X \times \mathbb{A}^m)$

Thm (Totaro)

In R,
$$[\mathrm{Sym}^n(X \times \mathbb{A}^m)] = [(\mathrm{Sym}^n X)]\,\mathbb{A}^{mn}$$

$$[\mathrm{Sym}^n \mathbb{A}^m] = [\mathbb{A}^{mn}]$$