

Reminder: The Grothendieck
Ring \mathbb{R}

Additive generators: $[X]$ ↖ varieties/
iso

Additive relations $[X] = [U] + [Z]$

Multiplication $[X \cdot Y] = [X] \cdot [Y]$

The motivic zeta function

$$Z_X(t) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in R[[t]]$$

$$Z: R \longrightarrow 1 + tR[[t]] \subset R[[t]]^{\times}$$

+ \rightsquigarrow \times

example: $k = \mathbb{F}_q$

Weil zeta function.

#: $Z_X(t) \xrightarrow{\text{Ex.}}$

$Z_X(t)$

$\in \mathbb{Z}[[t]]$

Question

2000
Kapranov

$Z_X(t)$ rational?

evidence:

\mathbb{Q}

curves.

Hodge structures.
True!

Chen '94

H^1

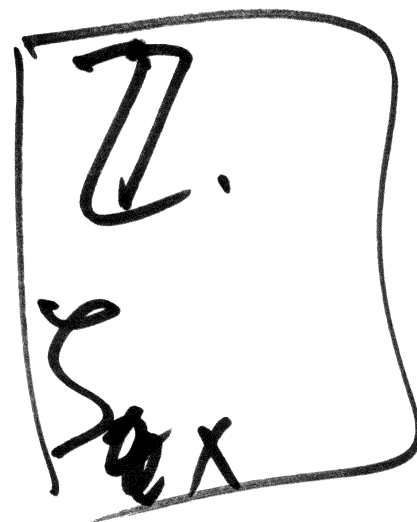
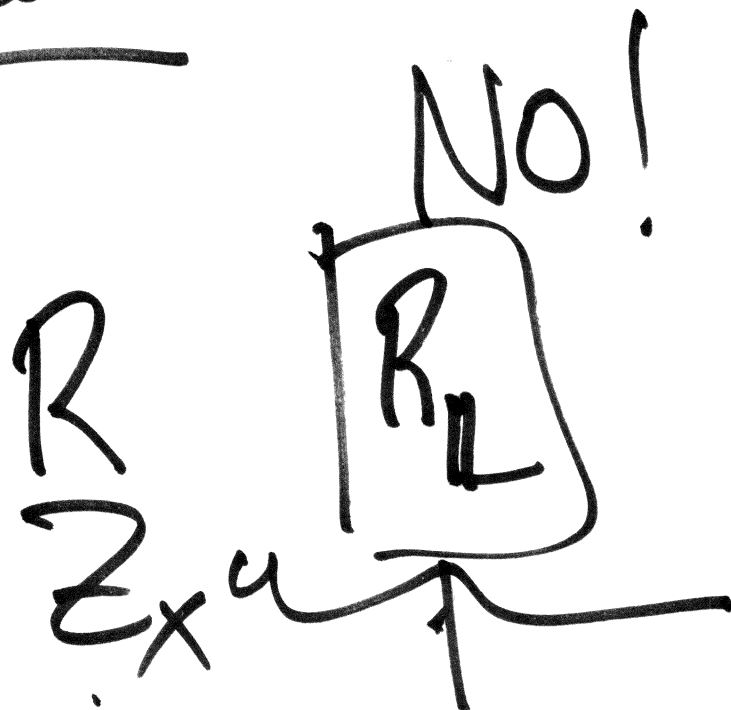
pt counting ✓

$$f \in \mathbb{C}[[t]] \\ = \frac{g}{h} \quad g, h \in \mathbb{C}(t).$$

$$fh = g$$

Theorem (Larsen-Lunts)

$k = \mathbb{F}_q$



$$k = \mathbb{C}$$

Thm

$$Z_X(t) =$$



$$\frac{(1-t)^{h_1} (1-t)^{h_3} \dots}{(1-t)^{h_0} (1-t)^{h_2} \dots}$$

Thm (Check)

...

Thm X complex variety

$$\sum_{n \geq 0} h^{p,q}(\mathrm{Sym}^n X) t^n x^p y^q (-z)^r$$

$$= \prod_{p,q,r} \left(\frac{1}{1 - x^p y^q z^r t} \right)^{(-1)^r h^{p,q}(H_c^r(X))}$$

Theorem ...

"cohomology-like"
 $H^i: \text{Var}_K \xrightarrow[\text{functor.}]{\text{contr.}} (VS + \text{more.})$
↘ field char 0.

$\oplus \otimes \wedge$

$$\chi(x) := \sum (-1)^i H^i(x) \in K(VS^+)$$

AXIOM: $H^i(x) = 0$ for $i \gg 0$

Theorem

$$\sum_{i=0}^{\infty} \chi(\text{Sym}^i X) t^i = \frac{1}{(1-t)^{\chi(X)}}$$

$$\in K(\cdot)[[t]]$$

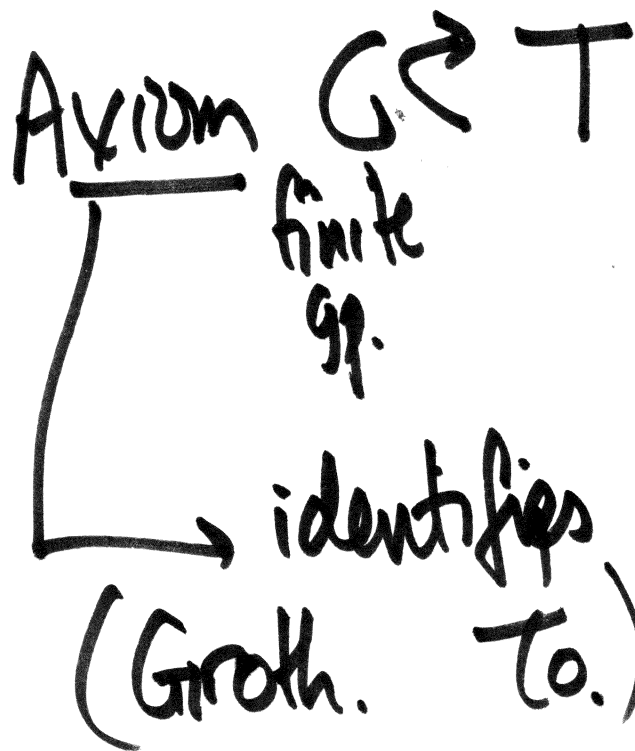
$$(1-t)^v = 1 - \binom{v}{1}t + \binom{v}{2}t^2$$

$$- \binom{v}{3}t^3 \dots = \pm \binom{v}{k}t^k$$

$$(1-t)^v (1-t)^w = (1-t)^{v \oplus w}$$

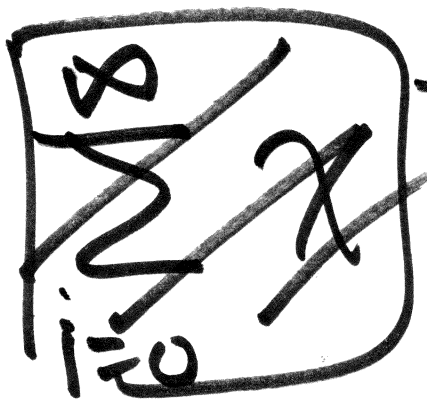
Ex $\frac{1}{(1-t)^V} = \sum_{n=0}^{\infty} \text{Sym}^n V t^n$

Axiom (Künneth) $H^n(X+Y) = \bigoplus_{i=0}^n H^i(X) \otimes H^{n-i}(Y)$



$\phi: T \rightarrow T/G$
 $\phi^*: H^i(T/G) \rightarrow H^i(T)$
 $H^i(T/G)$ with $(H^i(T))^G$

$$\begin{aligned}
 H^*(\text{Sym}^n X) & \xrightarrow{X^n / S_n} = \left(H^*(X^n) \right)^{S_n} \\
 & = \text{in terms of } H^*(X)
 \end{aligned}$$



Cor. Hodge structures
 \implies Cheah.

$$\begin{aligned}
 \text{Cor } k = \mathbb{F}_q \quad \# \quad F \curvearrowright H_{\text{ét}}^r(X, \mathbb{Q}_\ell) \\
 \sum_i (-1)^i \text{tr}(F | H^r(X, \mathbb{Q}_\ell))
 \end{aligned}$$

Larsen-Luntz:

$$\mu(X) = 1 + h^0(\Omega^1) t + h^0(\Omega^2) t^2 + \dots + h^0(\Omega^{\dim}) t^{\dim}$$

smooth
projective

$$\in 1 + t\mathbb{Z}[[t]]$$

$$\begin{aligned} \mu(L) &= 0 \\ &= \mu(P^1) - \mu(pt) \end{aligned}$$

Motivation

$$\sum_k(t) = \frac{P_1(t) \cdots P_{2d-1}(t)}{P_0(t) \cdots P_{2d}(t)} = \sum a_i t^i$$

$$a_i \sim \binom{2d}{i} \left(\frac{1-g^d}{g} t \right)^i \sim \text{Hase-Weil bounds}$$

Conj (Motive Sym Stabilization of IMSSP)

$$\frac{\text{Sym}^n X}{\mathbb{L}^{\dim X}} \rightarrow \mathbb{Z} \in \mathbb{R}_L$$

converges.

Evidence
[Stably rational
varieties

pt. counting ✓
Hodge structures ✓
curves ✓

Topology.

$p \in X$

$$\text{Sym}^n X \longrightarrow \text{Sym}^{n+1} X$$

effective Arnaud
Tripathy.

$$\text{Sym}^n X = \left(\prod K(\tilde{H}_i(X), n) \right).$$

Thm (Tripathy):
"Sym" étale realization.
commute.

$X \cong X \times \mathbb{A}^n$ "homotopiz"
 $\text{Sym}^n X \cong \text{Sym}^n (X \times \mathbb{A}^n)$

Thm (Totaro)

In \mathbb{R} ,

$$\left(\text{Sym}^n (X \times A^n) \right) \\ = \left[(\text{Sym}^n X) \right] A^{nn}$$

$$\left[\text{Sym}^n A^n \right] = \left[A^{nn} \right]$$