# Grothendieck ring of varieties.

$k$ field

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$R = R_k$

varieties $/k$
f. type schemes $/k$.

Additively:
gen. by $[X]$ $\quad X$ Variety / Iso.

relations: $\quad Z \xhookrightarrow{\text{cl.}} X \xhookleftarrow{\text{open}} U$

$[X] = [Z] + [U] \quad [X] = [X^{red}]$

Multiplication

$$[X]\cdot[Y] := [X \times Y]$$

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$R$ commutative ring

$$0 = [\emptyset]$$

$$1 = [\mathrm{pt}]$$

Speck

$$\mathbb{L} := [\mathbb{A}^1]$$

$$[\mathbb{P}^2] = \mathbb{L}^2 + \mathbb{L} + 1$$

$$\Big[\underset{\text{Zar. top.}}{\mathbb{P}^n\text{-bundle}} \;/\; X\Big] = [\mathbb{P}^n][X]$$

" "Classical" "Geometry" "

(finite CW complexes)

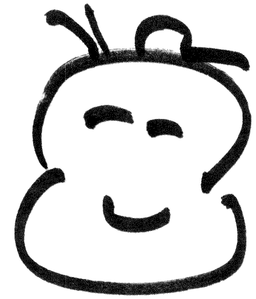
$$[\mathbb{R}] = [\mathbb{R}] + [\cancel{0}] + [\mathbb{R}] + 1$$

$$[\mathbb{R}^k] = (-1)^k.$$

$$X \longmapsto \chi_c(X) \cdot 1$$

$$0 = [\emptyset] = [S^1] = [S^3] = [S^1 \sqcup S^3]$$

$$v - e + f = 2 - g$$



BABY!

$R$ "ring of baby motives"

universal $\chi_c$

$k = \mathbb{F}_q$ $\qquad\qquad$ $R = ?$

$$X \longmapsto \mathbb{Z}.$$

#: count $\mathbb{F}_q$-pts = rat'l pts

e.g. $k = \mathbb{C}$.

$$\text{var } X \rightsquigarrow \chi_c(X)$$

$$\boxed{R_{\mathbb{C}} \longrightarrow \mathbb{Z}}$$

smooth complete

$$[\mathbb{CP}^2] = \mathbb{L}^2 + \mathbb{L} + 1$$

$$\chi(\mathbb{CP}^2) = 3$$

$$h^0 - h^1 + h^2 - h^3 + h^4.$$

Hodge structures

$$[\mathbb{C}^*] = \mathbb{L} - 1$$

$h^0 \qquad h^1$

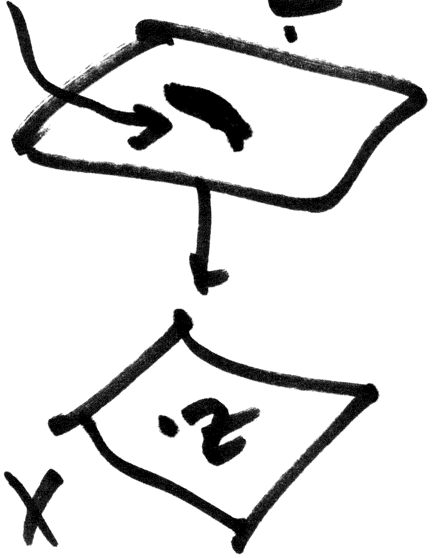
$= 1 \qquad = 1$

# Bittner's presentation in char 0

$R$ gen by $[X]$ ~~affine varieties~~ smooth

$\hookrightarrow$ projective smooth vars.

$E_Z$ $\quad Z \overset{\text{cl.}}{\hookrightarrow} X$ smooth.




$\mathrm{Bl}_Z X$

$X$

$$[\mathrm{Bl}_X Z] - [E_Y X]$$
$$= [X] - [Z]$$

**Theorem** (Bittner)

$R$ (additively) is: smooth proj

(Bitt rel'n.)

**Applications**

$a_0 : R \longrightarrow \mathbb{Z}$.

\# components for smooth projective

$h^i(X, \mathcal{O}_X)$ for smooth projective.

## Bittner duality.

$$R_{\mathbb{L}} = R\left[\frac{1}{\mathbb{L}}\right]$$

$$X \longrightarrow \frac{X}{\mathbb{L}^{\dim X}}$$

$1 = pt \longrightarrow pt = 1$

$p' \longrightarrow 1 + \frac{1}{t}$

$1 + t$

$A' = t \longrightarrow \frac{1}{t}.$

$R \longrightarrow R_{\mathbb{L}}$

injection if $\mathbb{L}$ not 0-divisor

Question / Conjecture / Speculation

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$\mathbb{L}$ not 0-div?

(False! L. Borisov).

$\underline{\text{Pf.}}$ of Bittner duality $\longrightarrow$ $c$ codim

$$[E_Z X] = \underbrace{[\mathbb{P}^{c-1}]}_{\mathbb{L}^{c-1}+\cdots}[Z]$$

$$(\mathbb{L}-1)[E_Z X] = (\mathbb{L}^c - 1)[Z]$$

$$\frac{[Bl_Z X]}{\mathbb{L}^{\dim X}} - \frac{[E_Z X]}{\mathbb{L}^{\dim X - 1}} = \frac{[X]}{\mathbb{L}^{\dim X}} - \frac{[Z]}{\mathbb{L}^{\dim X - c}}$$

## Applications.

$\mathbb{A}^m \times X \leftarrow - - - \rightarrow Y \times \mathbb{A}^n.$ $\boxed{\text{stably birat'l}}$

Var / stable birat'l

$\mathbb{Z}[SB]$ commutative semigroup

<u>Thm</u> (Larsen-Lunts)

$$\exists! \quad R \longrightarrow \mathbb{Z}[SB]$$

$$[X] \longrightarrow [X]$$

for sm. prj

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$$[Bl] - [E] = [X] - [Z]$$

$$(\mathbb{L}) \longrightarrow 0 \longrightarrow (\mathbb{L}) = \text{Kernel}$$

# Motivic Zeta Functions

(Kapranov 2000) $[e^X]$

**Def** $Z_X(t) = \sum [\mathrm{Sym}^n X] t^n$

$\in R[[t]]$

$$X = U_{op} \amalg Z_{cl.}$$

$$Z_X(t) = Z_U(t)\, Z_Z(t)$$

Can define:

$$Z_R(t) =$$

Example "" finite CW complexes.

$R = \mathbb{Z}$. $\chi_c$

$$Z_{pt}(t) = 1 + t + t^2 + t^3 + \cdots$$

$$= \boxed{\frac{1}{1-t}}$$

$$\therefore \quad \underset{\text{gps additive}}{R} \longrightarrow \underset{\text{mult}}{R[[t]]^*}$$

$1 + \cdots$

**Thm** (Macdonald '62)

$$\sum \chi_c[\mathrm{Sym}^n X]\, t^n = \frac{1}{(1-t)^{\chi_c(X)}}$$

**Pf.**

$$Z_{pt}(pt) = \frac{1}{1-t}$$

$$[X] = \chi_c(X)[pt]$$

$$Z_X(t) = \frac{1}{(1-t)^{\chi_c(X)}}$$

$k = \mathbb{F}_q.$

$$\# : Z_X(t) \longmapsto ?? \in$$

$$\in \mathbb{Z}[[t]]$$