Arizona Winter School 2015
Problem sessions
Curves on varieties & topics on surfaces
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1 Introduction

We give here a collection of questions related to the subjects of the Winter school. There are two types of exercises : some of them provide examples and some background and others give proofs (and necessary techniques) for some classical results mentioned during the lectures. In particular, we work out the proof of the classification of surfaces in characteristic zero. The exercises include the following topics : rational and unirational varieties, lines ?n varieties, exceptional locus, birational properties, divisors on surfaces, classification of surfaces, properties of Del Pezzo surfaces.

The exercises are of various difficulty, the background which should be enough for all the exercises is the Hartshorne's book «Algebraic geometry» [2]. The most difficult exercises are marked by (*) and (**), the additional hint requests are welcome.

In the appendix we collect a list of notions and recall some basic properties which are used in (some of) the exercises.

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2 Problems

Rational curves, rational and unirational varieties

1. Let $X \subset \mathbb{P}^n_k$ be a smooth quadric hypersurface. Show that $X(k) \neq \emptyset \Leftrightarrow X$ is birational to $\mathbb{P}^{n-1}_k$.
Hint: consider the space H of lines in $\mathbb{P}^n_k$ passing through $P \in X(k)$. Show that $H \simeq \mathbb{P}^{n-1}_k$. Show that the lines which are not tangent to X form an open $U \subset H$. Show that the map $U \to X$ that sends a line to its second intersection point with X is an isomorphism on its image.

2. Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a hypersurface of degree d. Prove that if $d \leq n-1$ then any point of X is contained in a line. For which values of d any two points could be joined by a chain of two lines?
Hint: Let $x \in X$ be a point, on can assume $x = (1 : 0 : \ldots : 0)$ (why?). If $y = (y_0 : \ldots : y_n) \in \mathbb{P}^n_{\mathbb{C}}$ is another point, write the equation of the line L_{xy} joining x and y. Write the conditions that $L_{xy} \subset X$. Show that there is always a solution y. For the second question fix two points x_1, x_2 and search for a point y such that $L_{x_1y}, L_{x_2y} \subset X$.

3. (*) Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism of degree d :
$$f(x:y) = (f_0(x:y) : \ldots : f_n(x:y)).$$
Let P_f be a point in some projective space $\mathbb{P}^N$ corresponding to the coefficients of f. Show that the condition that the polynomials f_i have a common factor gives a closed condition in $\mathbb{P}^N$. Deduce that there is a bijection between the set of maps f as above and points in some quasi-projective variety. What if we assume in addition that the image of f is included in some projective variety $X \subset \mathbb{P}^n$? (see [1] for the theory of these spaces of morphisms and applications to the study of rationally connected varieties).
Hint: Use Nullstellensatz to show that if $f_0, \ldots f_n$ have no common factor, then for some m there is a surjection of vector spaces
$$(k[x,y]_{m-d})^{n+1} \to k[x,y]_m, (g_0, \ldots, g_n) \mapsto \sum f_i g_i,$$
where $k[x,y]_m$ denotes the set of homogeneous polynomials of degree m. Deduce that some of $(m+1)$-minors of a matrix defining this map is non-zero. Then find a condition that $f_0, \ldots f_n$ have a common factor as intersection of (infinitely many) closed subsets.

4. Let X be a variety defined over an algebraically closed field k. Show that for any field extension K of k we have : X is rational $\Leftrightarrow X_K$ is rational.
Hint: For the implication $\Leftarrow$ show that one can assume that K is the function field of some variety Y over k. Show that $X \times Y$ is birational to $\mathbb{P}^n \times Y$, then deduce that X is rational.

Unirational varieties

We will show that for a unirational variety one can always find a dominant rational map $\phi : \mathbb{A}^n_k \dashrightarrow X$ with $n = dim\, X$:

5. Show that there exists $U \subset \mathbb{A}^n_k$ an open such that ϕ gives a morphism $U \to X$ with $n - dim\, X$ dimensional fibres.
Hint: a possibility is to use the flatness.

6. Assume that k is infinite. Show that there exist a point $u \in U(k)$ and a hyperplane H containing u such that $\phi|_H : H \dashrightarrow X$ is dominant. Conclude by induction.
Hint: Show that containing a fiber of $U \to X$ is a closed condition on the set of all hyperplanes H containing u.

7. Assume that k is finite. Take ℓ a prime, $\ell \neq char(k)$ and K the subfield of $\bar{k}$ formed as a union of all algebraic extensions of k of degree $\ell^s, s > 0$.

    (a) Show that $U(K)$ is not empty.

    (b) Let $u = (u_1, \dots u_n) \in U(K)$. Assuming that $k(u_n) \subset k(u_{n-1}) \dots \subset k(u_1)$, show that the ideal $I(u) \subset k[x_1, \dots x_n]$ is generated by polynomials of the form $x_n - P(x_1, \dots, x_{n-1})$ (i.e. by polynomials linear in x_n).
    Hint : Show that u_n can be expressed as a polynomial in u_1.

    (c) (*) Show that there exists a hypersurface $Z \subset \mathbb{A}^n_k$ defined by an equation $x_n - P(x_1, \dots, x_{n-1}) = 0$ such that $u \in Z(K)$ and the restriction $\phi|_Z : Z \dashrightarrow X$ is dominant. Conclude by induction.
    Hint: Use the previous exercises for the infinite field K.

Del Pezzo surfaces over an algebraically closed field

8. Let k be a field. Let $f : Y \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ be the blow up of at the intersection point of two lines $L_1 = (1:0) \times \mathbb{P}^1_k$ and $L_2 = \mathbb{P}^1_k \times (1:0)$.

   (a) Show that Y has two disjoint exceptional curves.
   Hint: compute L_i^2 and $(f^*L_i)^2$.

   (b) (*)What we get if we contract these curves?
   Hint (of a possible argument): Consider

   $$Z \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 = \{[(x:u),(y:v),(x_0:x_1:x_2)], x_0u - x_2x = 0, x_1v - x_2y = 0.\}$$

   let $\pi : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection map. Examine $\pi^{-1}([(1:0),(1:0)])$. Using the universal property of a blow up, show that one has a morphism $Z \to Y$. Let $Z \to \mathbb{P}^2$ be the projection of Z on $\mathbb{P}^2$. Let P_1, P_2 be the images of $\tilde{L}_1$ and $\tilde{L}_2$ and let W be the blow up of these points. Show that one has a map $Y \to W$ which is an isomorphism.

Let X be a Del Pezzo surface over an algebraically closed field k of characteristic zero.

9. Assume that $C \subset X$ is an irreducible smooth curve such that $C \cdot C < 0$. Prove that C is an exceptional curve and that $C \cdot C = -1$.
Hint: could an ample divisor have negative or zero intersection with an irreducible curve?

10. Assume that X is a blow up of $\mathbb{P}^2_k$ in r distinct closed points $P_1, \ldots, P_r$.

    (a) Show that $K_X^2 = 9 - r$.
    Hint: Use the formula for the canonical divisor of a blow up

    (b) Assume that three of the points $P_1, \ldots P_r$ lie on a line. Prove that X has an irreducible curve C such that $C \cdot C < -1$.
    Hint: if L is the line compute the self-intersection $\tilde{L} \cdot \tilde{L}$ where $\tilde{L}$ is the strict transform of L, same for the next questions.

    (c) Assume that six of the points $P_1, \ldots P_r$ lie on a conic. Prove that X has an irreducible curve C such that $C \cdot C < -1$.

    (d) (*) Assume that eight of the points $P_1, \ldots P_r$ lie on a cubic, which is singular at one of these points. Prove that X has an irreducible curve C such that $C \cdot C < -1$.

    (e) Deduce that if $-K_X$ is ample, then $0 \leq r \leq 8$ and the r points are in general position (meaning that no three points lie on a line, no six on a conic, and no eight on a cubic which is singular at one of these points).
    Hint: use the previous exercise.

11. Show that $h^1(X, mK_X) = 0$ for $m \geq 0$.
Hint: use Serre duality and Kodaira vanishing.

12. Show that $h^0(-mK_X) = \frac{m(m+1)}{2}d$ for all $m \geq 0$.
Hint: use Riemann-Roch.

13. Prove that a Del Pezzo surface is a rational surface.
Hint: use Castelnuovo criterion.

14. Deduce that if X is a Del Pezzo surface with $rk\, N(X) = 1$ then $X \simeq \mathbb{P}^2$.
Hint: prove that X has no exceptional curves, deduce that X is isomorphic to $\mathbb{P}^2$ or to one of the surfaces F_n - take a look to the appendix.

15. Assume that X has an exceptional curve C and $X \to Y$ a contraction of C. Show that Y is a Del Pezzo surface. What is the degree of Y?
Hint: use the formula for the canonical class of a blow up.

16. Show that a Del Pezzo surface with no exceptional curves is isomorphic to $\mathbb{P}^2_k$ or to $\mathbb{P}^1_k \times \mathbb{P}^1_k$. (one can use the classification of surfaces here : theorem 1 below).
Hint: in case X is a conic bundle over a curve C use that X is rational to deduce that $C \simeq \mathbb{P}^1_k$. Then use [2] V.2.9 et V.2.13 to find an exceptional curve if $X \neq \mathbb{P}^1_k \times \mathbb{P}^1_k$.

17. Show that there exists a Del Pezzo surface Y having no exceptional curves and a map $X \to Y$ which is a sequence of blowing ups.
Hint: proceed by induction controlling the degree.

18. Let X be a Del Pezzo surface of degree d. Show that X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or to a blow up of $\mathbb{P}^2_k$ at $9 - d$ points in general position.
Hint: use the question above and the question 8.

Del Pezzo surfaces : rationality

Let X be a Del Pezzo surface over a field k of characteristic zero.

19. Show that the degree d of a Del Pezzo X surface is at most nine.
    Hint: what changes if we consider $X_{\bar{k}}$?

20. If $d = 9$, show that $X(k) \neq 0 \Leftrightarrow X \simeq \mathbb{P}^2_k$.
    Hint: examine $X_{\bar{k}}$

21. Assume $d = 8$.

    (a) If $X(k) = \emptyset$, show that $X_{\bar{k}} = \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$.
    Hint: Show that $X_{\bar{k}} = \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ or $X_{\bar{k}}$ is a blow up of $\mathbb{P}^2_{\bar{k}}$ at one point. In the latter case show that the exceptional curve is defined over k.

    (b) If $X(k) \neq 0$, what are the possiblities for $X_{\bar{k}}$?

    (c) Assume $X(k) \neq 0$ and $X_{\bar{k}} = \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$. Find a divisor on $X_{\bar{k}}$ invariant under the Galois action. Does its class come from $\operatorname{Pic} X$? Could you use it to embed X in a projective space?
    Hint: Let L_1, L_2 defined as in exercise 8. *Write $K_{X_{\bar{k}}}$ in terms of L_1 and L_2. Deduce that $H = L_1 + L_2$ is invariant under the Galois action. Use the exact sequence* $\operatorname{Pic} X \to \operatorname{Pic} X_{\bar{k}} \to \operatorname{Br} k \to \operatorname{Br} X$ *which is split if $X(k) \neq \emptyset$. Show that H is base point free and that deg $H = 2$.*

    (d) Deduce that if $X(k) \neq 0$, then X is birational to $\mathbb{P}^2_k$.
    Hint : Use exercise 1.

22. Assume $d = 7$.

    (a) What are exceptional curves on $X_{\bar{k}}$?
    Hint: Show that $X_{\bar{k}}$ is a blow up of $\mathbb{P}^2_{\bar{k}}$ in two points. Use the strict transform $\tilde{L}$ of the line joining these two points.

    (b) Can one contract a curve on X?
    Hint: Show that $\tilde{L}$ is defined over k.

    (c) Could one have $X(k) = \emptyset$? X nonrational?
    Hint: Use exercise 2(d).

Remark. For $d = 5, 6$ one has $X(k) \neq \emptyset \Leftrightarrow X$ is birational to $\mathbb{P}^2_k$ (ask for more exercises! Or see [3]).

Divisors and positivity : preparation for the classification results

In this series of exercises we will be interested in various properties of divisors on surfaces. Let k be a field, X a smooth projective geometrically connected surface over k.

23. Show that if H is ample and $L \in \operatorname{Pic} X$, then for any $n > 0$ big enough $nH + L$ is ample. Deduce that $\operatorname{Pic} X$ is generated by classes of ample divisors.
Hint: use the cohomological characterization of ampleness.

24. (*) Show that if $L \in N(X)_{\mathbb{Q}}$ is nef and $H \in N(X)_{\mathbb{Q}}$ is ample, then for any $a \in \mathbb{Q}_{>0}$ one has $L + aH \in \operatorname{Amp}(X)$. Deduce that $L \cdot L \geq 0$.
Hint: Use Nakai-Moishezon criterion to show that there exists $a > 0$ such that $L + aH \in \operatorname{Amp}(X)$. Show that $L(L + aH) \geq 0$ and deduce that $L + \frac{a}{2}H$ is ample.

25. Describe $\overline{\operatorname{Ef}(X)}$ for the following complex surfaces :
    (a) $X = \mathbb{P}^2$,
    (b) $X = \mathbb{P}^1 \times \mathbb{P}^1$,
    (c) (**) X a $\mathbb{P}^1$-bundle over a smooth curve C of genus g (see [2] V.2 if needed).

26. (**) Let $L \in N(X)_{\mathbb{Q}}$. Show that $L \in \operatorname{Amp}(X) \Leftrightarrow L \cdot D > 0$ for any $D \in \overline{\operatorname{Ef}(X)}, D \neq 0$.
Hint : Assume $L \in \operatorname{Amp}(X)$.
    (a) Show that $\forall D \in \overline{\operatorname{Ef}(X)}$ one has $L \cdot D \geq 0$.
    (b) Assume that $\exists D \in \overline{\operatorname{Ef}(X)}$ such that $L \cdot D = 0$. Show that $\exists M \in N(X)_{\mathbb{R}}$ such that $(nL + M) \cdot D < 0 \, \forall \, n$.
    (c) Show that one can write $M = M_1 + M_2$ with $M_1 \in N(X)_{\mathbb{Q}}$ and $M_2 \in N(X)_{\mathbb{R}}$ "small enough" such that $nL + M_1$ is ample and $M_2 \cdot D \leq M \cdot D$.
    (d) Show that one gets a contradiction for M_2 "small enough". Deduce that $L \cdot D > 0 \, \forall D \in \overline{\operatorname{Ef}(X)}$.
    (e) Let $L \in \operatorname{Pic} X$ be such that $L \cdot D > 0 \, \forall \, D \in \overline{\operatorname{Ef}(X)}$. Show that $L \cdot C > 0$ for any irreducible curve C.
    (f) Let H be an ample divisor. Show that $\exists \epsilon > 0$ such that $L \cdot D \geq \epsilon H \cdot D \, \forall D \in \overline{\operatorname{Ef}(X)}$.
    (g) Deduce that $L - \epsilon H$ is nef and then that L is ample.

27. Show that $\operatorname{Amp}(X)$ is an open cone in $N(X)_{\mathbb{R}}$ with closure $\operatorname{Nef}(X)$.
Hint:

(a) *Use exercise 24 to show that* $\mathrm{Amp}\,(X)$ *is dense in* $\overline{\mathrm{Ef}\,(X)}$.

(b) *Let* $|\ \ |$ *be the restriction of any euclidean norm on* $N(X)_{\mathbb{R}}$. *Show that* $\exists\,\epsilon > 0$ *such that* $\forall D \in \overline{\mathrm{Ef}\,(X)}$, $|D| < 1 \Rightarrow H \cdot D \geq \epsilon$.

(c) *Show that for* D *small enough* $H + D$ *is ample.*

28. Let H be an ample divisor on X and let $L \in \mathrm{Pic}\, X$ such that $L \cdot L > 0$.

 (a) Show that $L \cdot H > 0 \Leftrightarrow H^0(X, nL) \neq 0$ for some $n > 0$ (one can use the Riemann-Roch theorem for this question).
 Hint: for $\Leftarrow$ *show that there exists an effective divisor linearly equivalent to* nL, *for* $\Rightarrow$ *show that* $(K - nL)H < 0$ *for* n *big enough, then deduce that* $H^0(K - nL) = 0$.

 (b) Show that for any $L \in N(X)_{\mathbb{Q}}$ one has $L \cdot H = 0 \Rightarrow L \cdot L \leq 0$.
 Hint: if $L^2 > 0$ *show that* $\exists n$ *such that* $H' = L + nH$ *is ample and* $L \cdot H' > 0$. *Use the previous exercise for* H'.

 (c) Deduce that if $L, M \in N(X)$ satisfy: $L \cdot L > 0$ and $M \cdot L = 0$, then $M \cdot M \leq 0$ and $M \cdot M = 0$ iff $M = 0$.
 Hint: show that in an appropriate base $(H, H^{\perp})$ *the intersection form could be written as* $(1, -1, \ldots - 1)$.

29. (**rationality theorem**) Let k be a field of characteristic zero. Assume that the canonical sheaf K_X is not nef. Show that for any invertible ample sheaf H

$$b(H) := \sup\{t \in \mathbb{R}, H + tK_X \text{ is nef}\}$$

is a rational number:

 (a) show that $b(H) \in \mathbb{Q}$ if K_X is proportional to H;

 (b) let $a, b \in \mathbb{N}$ such that $H + \frac{a-1}{b}K_X$ is ample. Express $h^0(bH + aK)$ as a polynomial in a, b (you may need to use the Kodaira vanishing theorem here);
 Hint: show first that $b(H)$ *is finite, then compute* $h^0(bH + aK_X) = h^0(bH + (a-1)K + K)$.

 (c) if $b(H) \notin \mathbb{Q}$, show that one can choose $a, b \in \mathbb{N}$ such that $bH + aK$ is effective, but not nef;
 Hint: Show that $\exists a, b$ *such that* $\frac{a}{b} > b(H)$ *but* $\frac{a-1}{b} < b(H)$ *and such that* $h^0(bH + aK_X) \neq 0$. *Note that the first condition does not change if we replace* (a, b) *by* (ka, kb) *and choose an appropriate* k.

 (d) Conclude.

Classification of surfaces in characteristic zero

This part provides a proof of the classification theorem for surfaces over a field of characteristic zero. We will also use the results from the exercises on «divisors and positivity». Let k be a field of characteristic zero and X a smooth projective geometrically connected surface over k.

The classification theorem we will establish says that X satisfies one of the following properties :

Theorem 1.

(i) K_X *is nef;*

(ii) $-K_X$ *is ample and* $rk\, N(X) = 1$*;*

(iii) $rk\, N(X) = 2$ *and there exists* Y *a smooth projective, geometrically connected curve and a morphism* $X \to Y$ *whose fibers are integral conics;*

(iv) X *has an exceptional curve.*

30. Give an example of a surface that satisfies (iii) and (iv).
Hint: Consider a blow up of $\mathbb{P}^2_k$ *in one point.*

31. Show that a surface X can only have one of the properties (i)-(iv), unless X satisfies (iii) and (iv) as in the previous question.
Hint: use that the rang of Pic *grows by* 1 *after a blow up.*

32. **Conic bundles**. Assume there exists a morphism $f : X \to Y$ with connected fibers and Y a smooth, projective, geometrically integral curve, such that for any component C of a fiber of f one has $C \cdot K_X < 0$.

(a) Show that a general fiber F of f is a smooth curve of (arithmetic) genus 0.
Hint: use the adjunction formula.

(b) Show that f has no multiple fibers.
Hint: Use Tsen's theorem.

(c) Assume that f has a fiber $F = \sum_{i=1}^{r} a_i C_i$ with $C_i \subset X$ an integral curve and $r \geq 2$. Show that X has an exceptional curve.
Hint: use that for any i *one has* $C_i \cdot F = 0$ *(why?)*

(d) Show that if X has no exceptional curves, then $rk\, N(X) = 2$.
Hint: Show that if $L \in \mathrm{Pic}\,(X)$ *is such that the restriction of* L *to the generic fiber is trivial then* L *is the class of a fiber.*

(e) Show that if X has no exceptional curves, then for any point $P \in Y$ the geometric fiber $F_{\overline{\kappa(P)}}$ is either isomorphic to a line or to a union of two lines, conjugated by the Galois action. Deduce that X is of type (iii).

33. Assume that X has no exceptional curves and that K_X is not nef. Show that if $rk\,N(X) = 1$, then $-K_X$ is ample.

34. Assume that X has no exceptional curves and that K_X is not nef and $rk\,N(X) \geq 2$. We will construct a morphism $X \to Y$ as in 32 :

    (a) Show that there exists an ample line bundle H not proportional to K_X.
    Hint: use exercise 23.

    (b) Let $b(H) = \frac{a}{b} \in \mathbb{Q}$ as in the rationality theorem (29) for H and let $L = bH + aK_X$. Show that $L \cdot L = 0$.
    Hint: show that $H + b(H)K$ is nef but not ample. Assuming that $L^2 > 0$ show that $\exists$ a curve C such that $L \cdot C \geq 0$. Deduce that C is an exceptional curve.

    (c) Show that $K \cdot L < 0$ and that there exists $m > 0$ with $h^0(X, mL) \geq 2$.
    Hint: show that $K - mL$ is not effective and use Riemann-Roch.

    (d) Let $L' = mL$. Let $S \in \operatorname{Pic} X$ be the class of a fixed part of L' and let M be such that $L' = M + S$. Prove that M is nef, $M \cdot M = 0$, $M \cdot S = 0$, $M \cdot K_X < 0$ and that $h^0(X, M) \geq 2$.
    Hint: show that the linear system $|M|$ has no fixed component. Deduce that M is nef. Use that $L^2 = 0$ to show that $M^2 = 0$ and $M \cdot S = 0$.

    (e) Let $\sum a_i C_i$ be an effective section of M. Show that $C_i \cdot K_X < 0$ for any i.
    Hint: show that $M \cdot C_i = 0\,\forall i$ and that $L' \cdot C_i = 0$. Deduce that $H \cdot C_i > 0$ and $K_X \cdot C_i < 0$.

    (f) Show that $H^0(X, M)$ is base point free. Let $X \to \mathbb{P}^N$ be a corresponding morphism and let Z be its image. Show that if $X \to Y \to Z$ is the Stein factorisation, then $f : X \to Y$ satisfies the conditions of 32 : it has connected fibers, Y is a smooth, projective, geometrically integral curve, such that for any component C of a fiber of f one has $C \cdot K_X < 0$.
    Hint: if $\sum n_i C_i$ is a section of M, show that $\exists$ Y_i a section of M which does not contain C_i. Show that $Y_i \cdot C_i = 0$ and that Y_i does not intersect C_i. Deduce that $H^0(X, M)$ is base point free.

35. Put all together to get the classification theorem.

Some birational properties

36. **Existence of rational points : Nishimura's lemma**:

    (a) Let k be a field and let X, Y be two varieties over k with Y proper over k. Let $f : X \dashrightarrow Y$ be a rational map. Assume that X has a smooth k-point. Prove that $Y(k)$ is not empty. (if $x \in X(k)$ is a smooth point, change X by $X' = Bl_x X$ and proceed by induction.)
    Hint: show that the rational map $X' \dashrightarrow Y$ is defined on an open subset of the exceptional divisor.

    (b) Deduce that the property «$X(k) \neq \emptyset$» is a birational invariant of smooth proper varieties over k.

    (c) (*) **A version with a group action**: Let k be an algebraically closed field, G be a linear (not necessarily connected) algebraic group over k, X, Y be two varieties equipped with an action of G with Y proper over k. Let $f : X \dashrightarrow Y$ be a G-equivariant rational map. Assume that any action of G on a projective space has a fixed point (as example of such a group, one can take a semi-direct product $G = U \rtimes A$ with U unipotent and A diagonalisable, see [A. Borel, Linear algebraic groups, I.4.8]). Prove that if X has a smooth point x fixed by G, then Y has a point fixed by G.
    Hint: Using the universal property of a blow up, show that the action of G lifts to a group action on $X' = Bl_x X$, having a fixed point on the exceptional divisor.

37. (*) **Exceptional locus**. Let $f : X \to Y$ be a birational morphism. The exceptional locus $E = Exc(f)$ is the locus of points $x \in X$ where the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is not an isomorphism.

    (a) Assume Y is normal and f is projective. Show that $f(E)$ is of codimension at least 2 in Y and that $E = f^{-1}(f(E))$.
    Hint : Using Zariski's Main theorem, show that a fiber containing at least two points is of dimension at least 1.

    (b) (**) Assume that Y is smooth. Prove that any component of E has codimension one in X (if $x \in X$, $y = f(x)$ and $t \in \mathfrak{m}_{X,x}$ not in $\mathcal{O}_{Y,y}$, write t as a difference of two effective divisors, i.e. $t = \frac{u}{v}$. For $Z \subset Y$ defined by $u = v = 0$ consider $f^{-1}(Z) \subset E$.)
    Hint : one can make a local computation as follows :

       i. *identify $k(X)$ and $k(Y)$ so that $\mathcal{O}_{Y,y}$ is a subring of $\mathcal{O}_{X,x}$;*
       ii. *show that $\exists t \in \mathfrak{m}_{X,x}$ but not in $\mathcal{O}_{Y,y}$, show that one can write t as a difference of two effective divisors, i.e. $t = \frac{u}{v}$ with $v \in \mathfrak{m}_{Y,y}$.*
       iii. *Consider $Z \subset Y$ defined by $u = v = 0$. What is the equation of $f^{-1}(Z)$?*
       iv. *Show that $y \in Z$ and that $f^{-1}(Z) \subset E$.*

3 Appendix : some background

Rationality properties.

- A variety X is rational if X is birational to a projective space.
- A variety X over a field k is **unirational** if there exists a dominant rational map $\phi : \mathbb{A}_k^n \dashrightarrow X$.
- A minimal rational surface X is isomorphic to $\mathbb{P}^2$ or to $F_n = \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), n \geq 0$ (minimal means that any dominant morphism $X \to Y$ to a smooth projective geometrically connected surface Y is an isomorphism.)

Some surfaces.

- A smooth projective geometrically connected surface X defined over a field k is a **Del Pezzo** surface if $-K_X$ is ample.
- The **degree** of a Del Pezzo surface X is the self-intersection number $K_X \cdot K_X$.

In what follows k is a field, X is a smooth projective geometrically connected surface over k.

Divisors and intersections (see [2] V).

- The Picard group $\operatorname{Pic} X$ is the group of invertible sheaves, up to isomorphism: this group is isomorphic to the group of divisors on X modulo linear equivalence. One has a bilinear symmetric intersection form $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$. One says that $L \in \operatorname{Pic} X$ is *numerically effective or nef* (resp. *trivial*) if for any (integral) curve $C \subset X$ one has $L \cdot C \geq 0$ (resp. $L \cdot C = 0$).
- A morphism $f : X \to Y$ of smooth projective surfaces induces the following maps on the Picard groups : $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ et $f_* : \operatorname{Pic} X \to \operatorname{Pic} Y$. One has the projection formula :

$$C \cdot f^* D = f_* C \cdot D,\ C \in \operatorname{Pic} X \in, D \in \operatorname{Pic} Y.$$

- The group $N(X)$ is the quotient of $\operatorname{Pic} X$ by the subgroup of numerically trivial divisor classes, the **Néron-Severi theorem** says that this group is a free group of finite type, so that one gets a nondegenerate pairing $N(X) \times N(X) \to \mathbb{Z}$. One denotes $N(X)_{\mathbb{Q}} = N(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, it is a convex rational cone (i.e. stable by addition and multiplication by $\mathbb{Q}_{\geq 0}$). One also introduces the convex real cone $N(X)_{\mathbb{R}} = N(X) \otimes_{\mathbb{Z}} \mathbb{R}$, endowed with a natural (real) topology.

- $\mathrm{Ef}\,(X) \subset N(X)_{\mathbb{Q}}$ is the subgroup generated by classes $r[C]$ where $r \geq 0$ a rational number and $C \subset X$ is an integral curve, $\overline{\mathrm{Ef}\,(X)} \subset N(X)_{\mathbb{R}}$ is the closure of $\mathrm{Ef}\,(X)$ in $N(X)_{\mathbb{R}}$.

- $\mathrm{Nef}\,(X) = \{\alpha[L], \alpha \in \mathbb{R}_{\geq 0}, L \text{ is nef } \} \subset N(X)_{\mathbb{R}}$.

- $\mathrm{Amp}\,(X) = \{r[H], r \in \mathbb{Q}_{\geq 0}, H \text{ is ample } \} \subset N(X)_{\mathbb{Q}}$. One has the following criterion for ampleness (**Nakai-Moishezon Criterion**) ([2] V.1.10) : L is ample if and only if $L \cdot L > 0$ and $L \cdot C > 0$ for any integral curve $C \subset X$.

- If $D \in \mathrm{Pic}\, X$, then one defines the complete linear system

  $$|D| = \{\text{effective divisors linearly equivalent to } D\} = (H^0(X, D) - \{0\})/k^*.$$

  (cf. [2] II.7.7) A point $x \in X$ is a base point of D if $x \in L$ for any $L \in |D|$. If D has no base points, one says that L is **base-point free**. In this case if $\{s_0, \dots s_N\}$ is a base of $H^0(X, D)$, one gets a morphism

  $$X \to \mathbb{P}^N, x \mapsto (s_0(x) : \dots : s_N(x).)$$

  The **fixed** part S of L is the greatest divisor contained in any $L \in |D|$.

Exceptional curves (see [2] V, [4].9).

- Let $f : Y \to X$ be a blowing-up of a closed point $x \in X$ and let $E \subset Y$ be the exceptional divisor. Let $C \subset X$ be a curve passing through x such that x is a smooth point of E. Let $\tilde{C}$ be the closure of $f^{-1}(C \setminus \{0\})$ in Y: we say that $\tilde{C}$ is the **strict transform** of C. Then

  $$E^2 = -1,\ K_X \cdot E = -1, C \cdot D = f^*C \cdot f^*D\ for\ C, D \in \mathrm{Pic}\, X,$$

  $K_Y = f^*K_X + E, f^*C = \tilde{C} + E.$

- An integral curve $C \subset X$ is **exceptional** if there exists a smooth projective geometrically connected surface Y, together with a birational morphism $f : X \to Y$, such that $f(C)$ is a closed point $c \in Y$ and f induces an isomorphism $X \setminus C \simeq Y \setminus c$. We then have that X is a blow up of Y at c.

  If $C \simeq \mathbb{P}^1_k$ and $C^2 = -1$ then C is exceptional. More generally, the **Castelnuovo criterion** says that an integral curve $C \subset X$ is exceptional if and only if $C \cdot C < 0$ and $C \cdot K_X < 0$ (see [4] 9.3.10).

- If k is algebraically closed, then any birational morphism $f : X \to Y$ of smooth projective surfaces over k factors as a sequence of blowing-ups of a closed point : $f : X = X_r \to X_{r-1} \to \dots \to X_1 \to X_0 = Y$.

Cohomological properties of surfaces (see [2] V).

- Adjunction formula: if $C \subset X$ a nonsingular curve of genus g then
$$2g - 2 = C \cdot (C + K_X).$$
- Serre duality: $h^i(X, L) = h^{2-i}(X, K_X - L), 0 \leq i \leq 2$.
- Riemann-Roch theorem: if D is a divisor on X, then
$$\chi(\mathcal{L}(D)) = \frac{1}{2} D \cdot (D - K_X) + \chi(\mathcal{O}_X).$$
- Kodaira vanishing theorem: if X is a smooth projective irreducible surface defined over a field k of characteristic zero, L is an ample divisor, then $H^1(X, -L) = H^1(X, K_X + L) = 0$.
- If X is a smooth projective irreducible surface over an algebraically closed field, then one has the following cohomological criterion of rationality, due to **Castelnuovo** : X is rational if and only if $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, 2K_X) = 0$.

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