Arizona Winter School 2015
Problem sessions
Curves on varieties & topics on surfaces
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1 Introduction

We give here a collection of questions related to the subjects of the Winter school. There are two types of exercises : some of them provide examples and some background and others give proofs (and necessary techniques) for some classical results mentioned during the lectures. In particular, we work out the proof of the classification of surfaces in characteristic zero. The exercises include the following topics : rational and unirational varieties, lines on varieties, exceptional locus, birational properties, divisors on surfaces, classification of surfaces, properties of Del Pezzo surfaces.

The exercises are of various difficulty, the background which should be enough for all the exercises is the Hartshorne's book «Algebraic geometry» [2]. The most difficult exercises are marked by (*) and (**), the additional hint requests are welcome.

In the appendix we collect a list of notions and recall some basic properties which are used in (some of) the exercises.

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2 Problems

Rational curves, rational and unirational varieties

1. Let $X \subset \mathbb{P}^n_k$ be a smooth quadric hypersurface. Show that $X(k) \neq \emptyset \Leftrightarrow X$ is birational to $\mathbb{P}^{n-1}_k$.

2. Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a hypersurface of degree d. Prove that if $d \leq n-1$ then any point of X is contained in a line. For which values of d any two points could be joined by a chain of two lines?

3. (*) Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism of degree d :
$$f(x:y) = (f_0(x:y) : \ldots : f_n(x:y)).$$
Let P_f be a point in some projective space $\mathbb{P}^N$ corresponding to the coefficients of f. Show that the condition that the polynomials f_i have a common factor gives a closed condition in $\mathbb{P}^N$. Deduce that there is a bijection between the set of maps f as above and points in some quasi-projective variety. What if we assume in addition that the image of f is included in some projective variety $X \subset \mathbb{P}^n$? (see [1] for the theory of these spaces of morphisms and applications to the study of rationally connected varieties).

4. Let X be a variety defined over an algebraically closed field k. Show that for any field extension K of k we have : X is rational $\Leftrightarrow X_K$ is rational.

Uniratinal varieties

We will show that for a unirational variety one can always find a dominant rational map $\phi : \mathbb{A}^n_k \dashrightarrow X$ with $n = dim\, X$:

5. Show that there exists $U \subset \mathbb{A}^n_k$ an open such that ϕ gives a morphism $U \to X$ with $n - dim\, X$ dimensional fibres.

6. Assume that k is infinite. Show that there exists a point $u \in U(k)$ and a hyperplane H containing u such that $\phi|_H : H \dashrightarrow X$ is dominant. Conclude by induction.

7. Assume that k is finite. Take ℓ a prime, $\ell \neq char(k)$ and K a subfield of $\bar{k}$ formed as a union of all algebraic extensions of k of degree $\ell^s, s > 0$.

   (a) Show that $U(K)$ is not empty.

   (b) Let $u = (u_1, \ldots u_n) \in U(K)$. Assuming that $k(u_n) \subset k(u_{n-1}) \ldots \subset k(u_1)$, show that the ideal $I(u) \subset k[x_1, \ldots x_n]$ is generated by polynomials of the form $x_n - P(x_1, \ldots, x_{n-1})$ (i.e. by polynomials linear in x_n).

   (c) (*) Show that there exists a hypersurface $Z \subset \mathbb{A}^n_k$ defined by an equation $x_n - P(x_1, \ldots, x_{n-1}) = 0$ such that $u \in Z(K)$ and the restriction $\phi|_Z : Z \dashrightarrow X$ is dominant. Conclude by induction.

Del Pezzo surfaces over an algebraically closed field

8. Let k be a field. Let Y be the blow up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at the intersection point of two lines $L_1 = (1:0) \times \mathbb{P}^1_k$ and $L_2 = \mathbb{P}^1_k \times (1:0)$.
   (a) Show that Y has two disjoint exceptional curves.
   (b) (*)What we get if we contract these curves?

Let X be a Del Pezzo surface over an algebraically closed field k of characteristic zero.

9. Assume that $C \subset X$ is an irreducible smooth curve such that $C \cdot C < 0$. Prove that C is an exceptional curve and that $C \cdot C = -1$.
10. Assume that X is a blow up of $\mathbb{P}^2_k$ in r distinct closed points $P_1, \dots, P_r$.
    (a) Show that $K_X^2 = 9 - r$.
    (b) Assume that three of the points $P_1, \dots P_r$ lie on a line. Prove that X has an irreducible curve C such that $C \cdot C < -1$.
    (c) Assume that six of the points $P_1, \dots P_r$ lie on a conic. Prove that X has an irreducible curve C such that $C \cdot C < -1$.
    (d) (*) Assume that eight of the points $P_1, \dots P_r$ lie on a cubic, which is singular at one of these points. Prove that X has an irreducible curve C such that $C \cdot C < -1$.
    (e) Deduce that if $-K_X$ is ample, then $0 \leq r \leq 8$ and the r points are in general position (meaning that no three points lie on a line, no six on a conic, and no eight on a cubic which is singular at one of these points).
11. Show that $h^1(X,_m K_X) = 0$ for $m \geq 0$.
12. Show that $h^0(-mK_X) = \frac{m(m+1)}{2} d$ for all $m \geq 0$.
13. Prove that a Del Pezzo surface is a rational surface.
14. Deduce that if X is a Del Pezzo surface with $rk\, N(X) = 1$ then $X \simeq \mathbb{P}^2$.
15. Assume that X has an exceptional curve C and $X \to Y$ a contraction of C. Show that Y is a Del Pezzo surface. What is the degree of Y?
16. Show that a Del Pezzo surface with no exceptional curves is isomorphic to $\mathbb{P}^2_k$ or to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ (one can use the classification of surfaces here : theorem 1 below).
17. Show that there exists a Del Pezzo surface Y having no exceptional curves and a map $X \to Y$ which is a sequence of blowing ups.

18. Let X be a Del Pezzo surface of degree d. Show that X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or to a blow up of $\mathbb{P}^2_k$ at $9-d$ points in general position.

Del Pezzo surfaces : rationality

Let X be a Del Pezzo surface over a field k of characteristic zero.

19. Show that the degree d of a Del Pezzo X surface is at most nine.

20. If $d = 9$, show that $X(k) \neq 0 \Leftrightarrow X \simeq \mathbb{P}^2_k$.

21. Assume $d = 8$.

    (a) If $X(k) = \emptyset$, show that $X_{\bar{k}} = \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$.

    (b) If $X(k) \neq 0$, what are the possiblities for $X_{\bar{k}}$?

    (c) Assume $X(k) \neq 0$ and $X_{\bar{k}} = \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$. Find a divisor on $X_{\bar{k}}$ invariant under the Galois action. Does its class come from $\operatorname{Pic} X$? Could you use it to embed X in a projective space?

    (d) Deduce that if $X(k) \neq 0$, then X is birational to $\mathbb{P}^2_k$.

22. Assume $d = 7$.

    (a) What are exceptional curves on $X_{\bar{k}}$?

    (b) Can one contract a curve on X?

    (c) Could one have $X(k) = \emptyset$? X nonrational?

Remark. For $d = 5, 6$ one has $X(k) \neq \emptyset \Leftrightarrow X$ is birational to $\mathbb{P}^2_k$ (ask for more exercises! Or see [3]).

Divisors and positivity : preparation for the classification results

In this series of exercises we will be interested in various properties of divisors on surfaces. Let k be a field, X a smooth projective geometrically connected surface over k.

23. Show that if H is ample and $L \in \operatorname{Pic} X$, then for any $n > 0$ big enough $nH + L$ is ample. Deduce that $\operatorname{Pic} X$ is generated by classes of ample divisors.

24. Show that if $L \in N(X)_{\mathbb{Q}}$ is nef and $H \in N(X)_{\mathbb{Q}}$ is ample, then for any $a \in \mathbb{Q}_{>0}$ one has $L + aH \in \operatorname{Amp}(X)$. Deduce that $L \cdot L \geq 0$.

25. Describe $\overline{\operatorname{Ef}(X)}$ for the following complex surfaces :
    (a) $X = \mathbb{P}^2$,
    (b) $X = \mathbb{P}^1 \times \mathbb{P}^1$,
    (c) X a $\mathbb{P}^1$-bundle over a smooth curve C of genus g (see [2] V.2 if needed).

26. (*) Let $L \in N(X)_{\mathbb{Q}}$. Show that $L \in \operatorname{Amp}(X) \Leftrightarrow L \cdot D > 0$ for any $D \in \overline{\operatorname{Ef}(X)}, D \neq 0$.

27. Show that $\operatorname{Amp}(X)$ is an open cone in $N(X)_{\mathbb{R}}$ with closure $\operatorname{Nef}(X)$.

28. Let H be an ample divisor on X and let $L \in \operatorname{Pic} X$ such that $L \cdot L > 0$.
    (a) Show that $L \cdot H > 0 \Leftrightarrow H^0(X, nL) \neq 0$ for some $n > 0$ (one can use the Riemann-Roch theorem for this question).
    (b) Show that for any $L \in N(X)_{\mathbb{Q}}$ one has $L \cdot H = 0 \Rightarrow L \cdot L \leq 0$.
    (c) Deduce that if $L, M \in N(X)$ satisfy: $L \cdot L > 0$ and $M \cdot L = 0$, then $M \cdot M \leq 0$ and $M \cdot M = 0$ iff $M = 0$.

29. (**rationality theorem**) Let k be a field of characteristic zero. Assume that the canonical sheaf K_X is not nef. Show that for any invertible ample sheaf H
$$b(H) := \sup\{t \in \mathbb{R}, H + tK_X \text{ is nef}\}$$
is a rational number:
    (a) show that $b(H) \in \mathbb{Q}$ if K_X is proportional to H;
    (b) let $a, b \in \mathbb{N}$ such that $H + \frac{a-1}{b} K_X$ is ample. Express $h^0(bH + aK)$ as a polynomial in a, b (you may need to use the Kodaira vanishing theorem here);
    (c) if $b(H) \notin \mathbb{Q}$, show that one can choose $a, b \in \mathbb{N}$ such that $bH + aK$ is effective, but not nef;
    (d) Conclude.

Classification of surfaces in characteristic zero

This part provides a proof of the classification theorem for surfaces over a field of characteristic zero. We will also use the results from the exercises on «divisors and positivity». Let k be a field of characteristic zero and X a smooth projective geometrically connected surface over k.

The classification theorem we will establish says that X satisfies one of the following properties :

Theorem 1.

(i) K_X *is nef;*

(ii) $-K_X$ *is ample and* $rk\, N(X) = 1$*;*

(iii) $rk\, N(X) = 2$ *and there exists* Y *a smooth projective, geometrically connected curve and a morphism* $X \to Y$ *whose fibers are integral conics;*

(iv) X *has an exceptional curve.*

30. Give an example of a surface that satisfies (iii) and (iv).

31. Show that a surface X can only have one of the properties (i)-(iv), unless X satisfies (iii) and (iv) as in the previous question.

32. **Conic bundles**. Assume there exists a morphism $f : X \to Y$ with connected fibers and Y a smooth, projective, geometrically integral curve, such that for any component C of a fiber of f one has $C \cdot K_X < 0$.
    - (a) Show that a general fiber F of f is a smooth curve of (arithmetic) genus 0.
    - (b) Show that f has no multiple fibers.
    - (c) Assume that f has a fiber $F = \sum_{i=1}^{r} a_i C_i$ with $C_i \subset X$ an integral curve and $r \geq 2$. Show that X has an exceptional curve.
    - (d) Show that if X has no exceptional curves, then $rk\, N(X) = 2$.
    - (e) Show that if X has no exceptional curves, then for any point $P \in Y$ the geometric fiber $F_{\overline{\kappa(P)}}$ is either isomorphic to a line or to a union of two lines, conjugated by the Galois action. Deduce that X is of type (iii).

33. Assume that X has no exceptional curves and that K_X is not nef. Show that if $rk\, N(X) = 1$, then $-K_X$ is ample.

34. Assume that X has no exceptional curves and that K_X is not nef and $rk\, N(X) \geq 2$. We will construct a morphism $X \to Y$ as in 32 :
    - (a) Show that there exists an ample line bundle H not proportional to K_X.

(b) Let $b(H) = \frac{a}{b} \in \mathbb{Q}$ as in the rationality theorem (29) for H and let $L = bH + aK_X$. Show that $L \cdot L = 0$.

(c) Show that $K \cdot L < 0$ and that there exists $m > 0$ with $h^0(X, mL) \geq 2$.

(d) Let $L' = mL$. Let $S \in \operatorname{Pic} X$ be the class of a fixed part of L' and let M be such that $L' = M + S$. Prove that M is nef, $M \cdot M = 0$, $M \cdot S = 0$, $M \cdot K_X < 0$ and that $h^0(X, M) \geq 2$.

(e) Let $\sum a_i C_i$ be an effective section of M. Show that $C_i \cdot K_X < 0$ for any i.

(f) Show that $H^0(X, M)$ is base point free. Let $X \to \mathbb{P}^N$ be a corresponding morphism and let Z be its image. Show that if $X \to Y \to Z$ is the Stein factorisation, then $f : X \to Y$ satisfies the conditions of 32 : it has connected fibers, Y is a smooth, projective, geometrically integral curve, such that for any component C of a fiber of f one has $C \cdot K_X < 0$.

35. Put all together to get the classification theorem.

Some birational properties

36. **Existence of rational points : Nishimura's lemma**:

    (a) Let k be a field and let X, Y be two varieties over k with Y proper over k. Let $f : X \dashrightarrow Y$ be a rational map. Assume that X has a smooth k-point. Prove that $Y(k)$ is not empty. (if $x \in X(k)$ is a smooth point, change X by $X' = Bl_x X$ and proceed by induction.)

    (b) Deduce that the property «$X(k) \neq \emptyset$» is a birational invariant of smooth proper varieties over k.

    (c) (*) **A version with a group action**: Let k be an algebraically closed field, G be a linear (not necessarily connected) algebraic group over k, X, Y be two varieties equipped with an action of G with Y proper over k. Let $f : X \dashrightarrow Y$ be a G-equivariant rational map. Assume that any action of G on a projective space has a fixed point (as example of such a group, one can take a semi-direct product $G = U \rtimes A$ with U unipotent and A diagonalisable, see [A. Borel, Linear algebraic groups, I.4.8]). Prove that if X has a smooth point x fixed by G, then Y has a point fixed by G.

37. (*) **Exceptional locus**. Let $f : X \to Y$ be a birational morphism. The exceptional locus $E = Exc(f)$ is the locus of points $x \in X$ where the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is not an isomorphism.

    (a) Assume Y is normal and f is projective. Show that $f(E)$ is of codimension at least 2 in Y and that $E = f^{-1}(f(E))$.

    (b) (**) Assume that Y is smooth. Prove that any component of E has codimension one in X (if $x \in X$, $y = f(x)$ and $t \in \mathfrak{m}_{X,x}$ not in $\mathcal{O}_{Y,y}$, write t as a difference of two effective divisors, i.e. $t = \frac{u}{v}$. For $Z \subset Y$ defined by $u = v = 0$ consider $f^{-1}(Z) \subset E$.)

3 Appendix : some background

Rationality properties.

- A variety X is rational if X is birational to a projective space.
- A variety X over a field k is **unirational** if there exists a dominant rational map $\phi : \mathbb{A}^n_k \dashrightarrow X$.
- A minimal rational surface X is isomorphic to $\mathbb{P}^2$ or to $F_n = \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), n \geq 0$ (minimal means that any dominant morphism $X \to Y$ to a smooth projective geometrically connected surface Y is an isomorphism.)

Some surfaces.

- A smooth projective geometrically connected surface X defined over a field k is a **Del Pezzo** surface if $-K_X$ is ample.
- The **degree** of a Del Pezzo surface X is the self-intersection number $K_X \cdot K_X$.

In what follows k is a field, X is a smooth projective geometrically connected surface over k.

Divisors and intersections (see [2] V).

- The Picard group $\operatorname{Pic} X$ is the group of invertible sheaves, up to isomorphism: this group is isomorphic to the group of divisors on X modulo linear equivalence. One has a bilinear symmetric intersection form $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$. One says that $L \in \operatorname{Pic} X$ is *numerically effective or nef* (resp. *trivial*) if for any (integral) curve $C \subset X$ one has $L \cdot C \geq 0$ (resp. $L \cdot C = 0$).
- A morphism $f : X \to Y$ of smooth projective surfaces induces the following maps on the Picard groups : $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ et $f_* : \operatorname{Pic} X \to \operatorname{Pic} Y$. One has the projection formula :

$$C \cdot f^* D = f_* C \cdot D, \ C \in \operatorname{Pic} X \in, D \in \operatorname{Pic} Y.$$

- The group $N(X)$ is the quotient of $\operatorname{Pic} X$ by the subgroup of numerically trivial divisor classes, the **Néron-Severi theorem** says that this group is a free group of finite type, so that one gets a nondegenerate pairing $N(X) \times N(X) \to \mathbb{Z}$. One denotes $N(X)_{\mathbb{Q}} = N(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, it is a convex rational cone (i.e. stable by addition and multiplication by $\mathbb{Q}_{\geq 0}$). One also introduces the convex real cone $N(X)_{\mathbb{R}} = N(X) \otimes_{\mathbb{Z}} \mathbb{R}$, endowed with a natural (real) topology.

- $\mathrm{Ef}\,(X) \subset N(X)_{\mathbb{Q}}$ is the subgroup generated by classes $r[C]$ where $r \geq 0$ a rational number and $C \subset X$ is an integral curve, $\overline{\mathrm{Ef}\,(X)} \subset N(X)_{\mathbb{R}}$ is the closure of $\mathrm{Ef}\,(X)$ in $N(X)_{\mathbb{R}}$.

- $\mathrm{Nef}\,(X) = \{\alpha[L], \alpha \in \mathbb{R}_{\geq 0}, L \text{ is nef }\} \subset N(X)_{\mathbb{R}}$.

- $\mathrm{Amp}\,(X) = \{r[H], r \in \mathbb{Q}_{\geq 0}, H \text{ is ample }\} \subset N(X)_{\mathbb{Q}}$. One has the following criterion for ampleness (**Nakai-Moishezon Criterion**) ([2] V.1.10) : L is ample if and only if $L \cdot L > 0$ and $L \cdot C > 0$ for any integral curve $C \subset X$.

- If $D \in \mathrm{Pic}\,X$, then one defines the complete linear system

  $$|D| = \{\text{effective divisors linearly equivalent to } D\} = (H^0(X, D) - \{0\})/k^*.$$

  (cf. [2] II.7.7) A point $x \in X$ is a base point of D if $x \in L$ for any $L \in |D|$. If D has no base points, one says that L is **base-point free**. In this case if $\{s_0, \ldots s_N\}$ is a base of $H^0(X, D)$, one gets a morphism

  $$X \to \mathbb{P}^N, x \mapsto (s_0(x) : \ldots : s_N(x).)$$

  The **fixed** part S of L is the greatest divisor contained in any $L \in |D|$.

Exceptional curves (see [2] V, [4].9).

- Let $f : Y \to X$ be a blowing-up of a closed point $x \in X$ and let $E \subset Y$ be the exceptional divisor. Let $C \subset X$ be a curve passing through x such that x is a smooth point of E. Let $\tilde{C}$ be the closure of $f^{-1}(C \setminus \{0\})$ in Y: we say that $\tilde{C}$ is the **strict transform** of C. Then

  $$E^2 = -1,\ K_X \cdot E = -1, C \cdot D = f^*C \cdot f^*D \ for\ C, D \in \mathrm{Pic}\,X,$$

  $K_Y = f^*K_X + E, f^*C = \tilde{C} + E$.

- An integral curve $C \subset X$ is **exceptional** if there exists a smooth projective geometrically connected surface Y, together with a birational morphism $f : X \to Y$, such that $f(C)$ is a closed point $c \in Y$ and f induces an isomorphism $X \setminus C \simeq Y \setminus c$. We then have that X is a blow up of Y at c.

  If $C \simeq \mathbb{P}^1_k$ and $C^2 = -1$ then C is exceptional. More generally, the **Castelnuovo criterion** says that an integral curve $C \subset X$ is exceptional if and only if $C \cdot C < 0$ and $C \cdot K_X < 0$ (see [4] 9.3.10).

- If k is algebraically closed, then any birational morphism $f : X \to Y$ of smooth projective surfaces over k factors as a sequence of blowing-ups of a closed point : $f : X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = Y$.

Cohomological properties of surfaces (see [2] V).

- Adjunction formula: if $C \subset X$ a nonsingular curve of genus g then

$$2g - 2 = C \cdot (C + K_X).$$

- Serre duality: $h^i(X, L) = h^{2-i}(X, K_X - L), 0 \leq i \leq 2$.

- Riemann-Roch theorem: if D is a divisor on X, then

$$\chi(\mathcal{L}(D)) = \frac{1}{2} D \cdot (D - K_X) + \chi(\mathcal{O}_X).$$

- Kodaira vanishing theorem: if X is a smooth projective irreducible surface defined over a field k of characteristic zero, L is an ample divisor, then $H^1(X, -L) = H^1(X, K_X + L) = 0$.

- If X is a smooth projective irreducible surface over an algebraically closed field, then one has the following cohomological criterion of rationality, due to **Castelnuovo** : X is rational if and only if $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, 2K_X) = 0$.

References

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