Thm$^{(*)}$ $X \longrightarrow \mathbb{P}^1_k$ <u>fibration</u> (X_η geom. int.

<u>Assume</u>: $\forall\, m \in \mathbb{P}^{1\,(1)}_k$ $\exists\, Y \subset X_m / k(m)$

component of multiplicity <u>one</u> and

the alg closure of $k(m)$ in $k(Y)$ is abelian

Abelian <u>splitting</u>

Assume <u>Schinzel</u> (H')

<u>Then</u>: If $X(\mathbb{A}_k)^{Br_{vert}(X)} \neq \emptyset$ then

$\exists\, t_0 \in \mathbb{P}^1(k)$ X_{t_0} smooth

and $X_{t_0}(\mathbb{A}_k) \neq \emptyset$.

Known if $d \leq 2$

Theorem (L. Matthiesen)

$\lfloor$ $k = \mathbb{Q}$, hyp (HW) is a theorem.

Theorem (H-W) Modulo (HW) hyp.

$X \to \mathbb{P}^1_k$ fibration in rat. conn. varieties

$X(\mathbb{A}_k)^{Br} \neq \emptyset$

$\Longrightarrow \exists\, t_0 \in \mathbb{P}^1_k$ with X_{t_0} smooth

and $\left(X_{t_0}(\mathbb{A}_k)\right)^{Br X} \neq \emptyset$

$\to$ If $k = \mathbb{Q}$ unconditional theorem

<u>Fn example</u>: this handles

$$N_{K/\mathbb{Q}}(\xi) = \prod_{i=1}^{d}(t - e_i) \neq 0 \quad U$$

<u>Thm</u> any field exten

$U \subset X$ $\qquad$ $\underline{\overline{X(\mathbb{Q})}^{top} = X(\mathbb{A}_{\mathbb{Q}})^{Br X}}$

X/k zero-cycle $\sum n_i P_i$

$n_i \in \mathbb{Z}$ $n_i = 0$ for almost all i

P_i is a closed point of X.

P closed point $k(P)$ residue field $[k(P):k]<\infty$

$$\deg_k\left(\sum n_i P_i\right) := \sum n_i [k(P_i):k] \in \mathbb{Z}.$$

<u>Pb</u>. given a k-variety is $X(k) \neq \emptyset$.

↳ is there a zero-cycle $z \in Z_0(X)$ of degree 1 ?

$\sum n_i P_i$ $\sum n_i (k(P_i):k) = 1$

X/k projective

$C \hookrightarrow X$ $\qquad f \in k(C)^*$

curve $\qquad \mathrm{div}_C(f) \in Z_0(C) \to Z_0(X)$

these span the gp of zero-cycles rat. ~ 0

quotient $\qquad CH_0(X) := Z_0(X)/\mathrm{rat.}$

$$\cancel{Z}_0(X) \times \mathrm{Br}\,X \longrightarrow \mathrm{Br}\,k$$

$$\sum n_i P_i \;,\; \alpha \longmapsto \sum n_i \,\mathrm{Cores}_{k(P_i)/k}(\alpha(P_i))$$

if X is projective, induces a pairing

$$CH_0(X) \times \mathrm{Br}\,X \longrightarrow \mathrm{Br}\,k$$

k number field X/k proj:

$$CH_0(X) \longrightarrow \prod_{v\in\Omega} CH_0(X\times_k k_v) \longrightarrow \mathrm{Hom}(\mathrm{Br}\,X, \mathbb{Q}/\mathbb{Z})$$

Complex

$$\begin{tikzcd} \mathrm{Br}\,k \arrow[r] \arrow[rr, bend right, "0"'] & \bigoplus \mathrm{Br}\,k_v \arrow[r] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

$$A \to \hat{A} = \varprojlim A/n$$

Conj.

(E) $$\widehat{CH_0(X)} \longrightarrow {\prod_{v\in\Omega}}' \widehat{CH_0(X\times_k k_v)} \longrightarrow \mathrm{Hom}(\mathrm{Br}\,X, \mathbb{Q}/\mathbb{Z})$$

is an exact sequence

$'$ ~ archimedean places.

(→ Sansuc, Kato, Saito)

1980's.

If $\dim X = 1$ $\quad$ $|Ш(J(X))| < \infty \longrightarrow$ the conjecture.

Salberger 1988

Theorem. This conjecture holds for any conic bundle $/\mathbb{P}^n_k$

$(\mathrm{Conj})(E) \Rightarrow \mathrm{Conj}(E_1)$: If $\exists\, (z_v) \in Z_0^1(X_{k_v})$

s.t. $\forall \alpha \in Br(X)$ $\sum \alpha(z_v) = 0 \in \mathbb{Q}/\mathbb{Z}$

$\Longrightarrow \exists$ zero-cycle of degree one

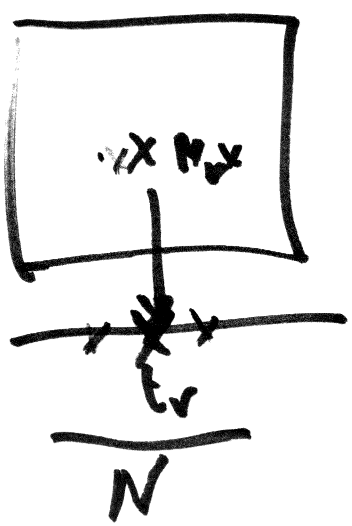
Salberger's pf in a special case

$U \quad y^2 - a z^2 = \underline{P(t)} \neq 0$ $\quad \deg = d.$

$a \in k^*$, $P(t) \in k[t]$ irreducible

Assume $\prod_{v \in S} U(k_v) \neq \emptyset$

S = obvious finite set of bad places



$v \in S$

$\boxed{\text{Fix } N > d}$

- $G_v(t) \in k_v[t]$ split $\prod (t - t_v^i)$
- $v_0 \notin S$ $\quad G_{v_0}(t)$ irreducible $\in k_{v_0}[t]$

$(a \in k_{v_0}^{\times 2})$

$$G_v(t) = P(t)\,Q_v(t) + R_v(t) \qquad d^\circ R_v < d^\circ P$$

$K = k[t]/P(t)$ $\qquad$ $R_v(t) \longrightarrow \xi_v \in K_v$.

<u>Dirichlet</u> $\longrightarrow$ $\xi \in K^\times$ $\qquad |\xi - \xi_v|_v < \varepsilon$

and $(\xi) = \underset{\substack{\uparrow \\ \text{unique prime of degree 1 over } k}}{\mathfrak{f}} \underbrace{\prod \mathfrak{q}^{n_\mathfrak{q}}}_{\text{above } S}$

. ξ lifts to an $R(t) \in k[t]$

fix $v_1 \notin S \cup v_0$ $\qquad a \in k_{v_1}^{\times 2}$

then use strong <u>approximat</u> to find

| $Q(t) \in k[t]$ monic very close to each $Q_v(t)$ for $v \in S$
| integral away from $S \cup v_0 \cup v_1$

$G(t) = P(t)\,Q(t) + R(t)$ irreducible

Consider the closed pt $\underbrace{\text{pt}}_{m}$ of $\mathbb{P}^1$ defined by $G(t)$.

$y^2 - a z^2 = P(t)$

$y^2 - a z^2 = \rho \;/\; k(m)$

where ρ = class of $P(t)$

<u>Claim</u> $y^2 - a z^2 = \rho / k(m)$ has a $k(m)$-pt.

<u>Pf.</u> If $v \notin S \cup v_0 \cup v_1$ ok by good reduction!

$v \in v_0$ or v_1 $\quad a \in k_v^{\times 2}$

If $v \in S$ there is a k_v-pt by approx.

<u>Thm</u> (Harpaz and Wittenberg) $\quad k = \#$ field

Let $X \longrightarrow \mathbb{P}^1_k$ be a fibration into rationally connected varieties

If $\forall\, m \in \mathbb{P}^1_k$ X_m smooth (E) holds for X_m

then (E) holds for X.

≤ 1998 | 2014

$X \to \mathbb{P}^1$ <u>R.C. fibres</u>

Rat. pts.

Schinzel + reciprocity + formal lemma $\longrightarrow$ $X \downarrow \mathbb{P}^1_k$ with abelian [illegible]

hyp (HW) + formal lemma for torsors under tori $\longrightarrow$ $X \downarrow \mathbb{P}^1_k$

~~Reciprocity~~

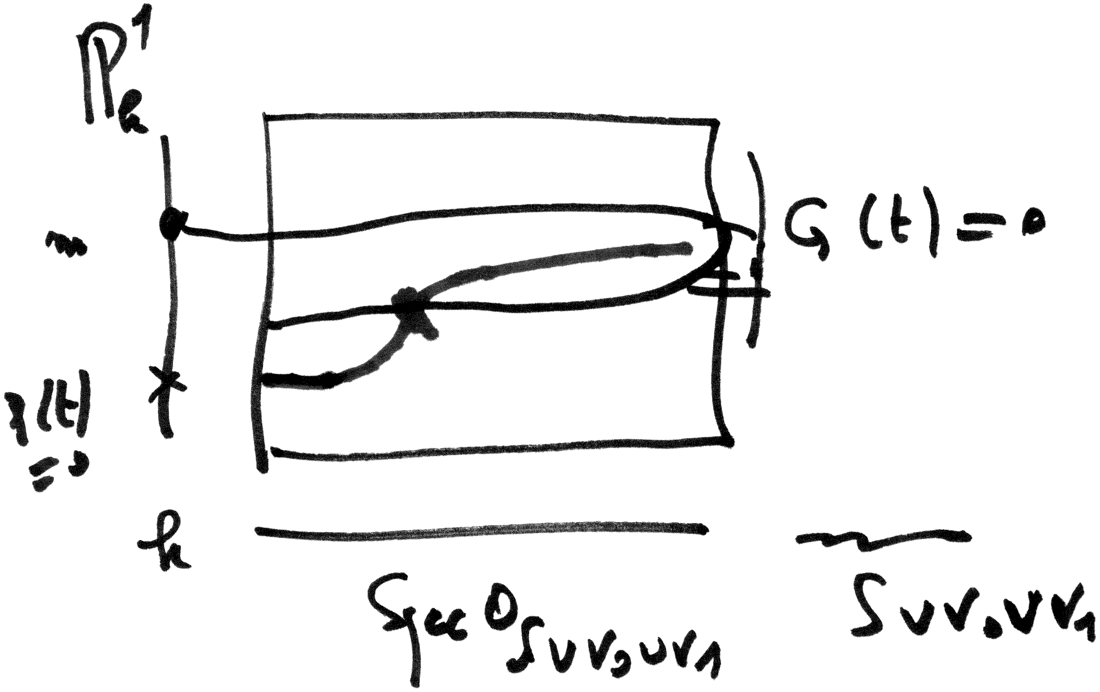
<u>**Zero-cycles**</u>

reciprocity + formal lemma <u>Salberger's trick</u> $\longrightarrow$ $X \downarrow \mathbb{P}^1_k$ with abelian [illegible]

(E) for X if HP + WA w. the fibres

<u>easy (HW) lemma</u> + ___ + elaborate version of Salberger's trick $\longrightarrow$ $X \downarrow \mathbb{P}^1_k$

(E) for X if known for the <u>fibres</u>



$\Rightarrow$ $y^2 - a z^2 = P$ has pts in all completions of k except possibly one k_v

$\Rightarrow$ has solution of degree N.

$\Rightarrow$ X has a $k(m)$-pt

but N was arbitrary ? $\Rightarrow$ $\exists$ zero-cycle of degree one

Was extended to $X \longrightarrow \mathbb{P}^1_k$

with (1) abelian splitting condition

(2) Fibers over any closed pt satisfy HP + WA

The easy HW lemma

K/k finite extension $\quad S \subset \Omega$

$v \in S \quad \xi_v \in k_v^\times \cap NK_v^\times \quad \varepsilon > 0$

$\Longrightarrow \exists\, \xi \in k^\times \quad |\xi - \xi_v|_v < \varepsilon \quad v \in S$

and $\forall v \notin S \quad v(\xi) = 0$

or

$\exists\, w$ of K of degree 1 over v

moreover if v_0 is split in K

may take ξ integral away from v_0

Pf only uses strong approximation.

Corollary (th. of hw)

$X \xrightarrow{f} \mathbb{P}^1_k$ dominant

and $X_\eta \sim$ a homogeneous space of $k(\mathbb{P}^1)$-birational a $k(\mathbb{P}^1)$-linear alg

g_p with connected stabilizer

then (E) holds for X

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ws! If $X \supset E$ E h.sp. of G/k connected linear)
$/k$

$\Rightarrow$ log (E) holds for X

(Yagi Liang)