Thm$^{(*)}$ $X \longrightarrow \mathbb{P}^1_k$ fibration (X_η geom. int.

Assume: $\forall\, m \in \mathbb{P}^{1\,(1)}_k$ $\exists\, Y \subset X_m / k(m)$

component of multiplicity one and

Abelian splitting

the alg closure of $k(m)$ in $k(Y)$ is abelian

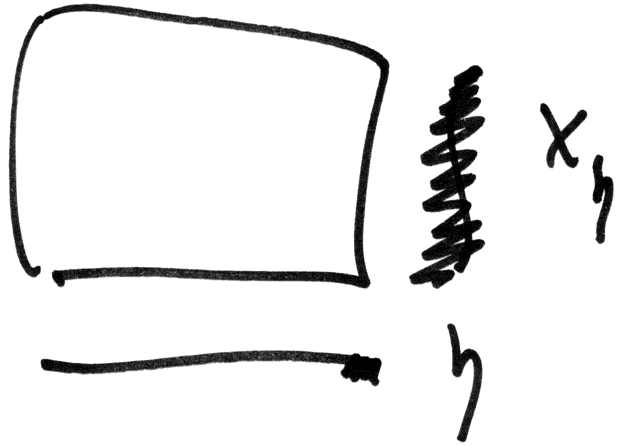
Assume Schinzel (H')

Then: If $X(\mathbb{A}_k)^{Br_{vert}(X)} \neq \emptyset$ then

$\exists\, t_0 \in \mathbb{P}^1(k)$ X_{t_0} smooth

and $X_{t_0}(\mathbb{A}_k) \neq \emptyset$.

$$\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1_k \end{array} \qquad Br_{vert}(X) = \{ \alpha \in Br X \quad \alpha|X_\eta \in Br\, k(\mathbb{P}^1) \}$$



Cor of Thm*: If moreover the fibre satisfy

HI + WA

then $X(k) \neq \emptyset$ $\quad \overline{X(k)}^{top} = X(\mathbb{A}_k)^{Br_{vert} X}$

Thm: $k = \mathbb{Q}$ $\quad X \xrightarrow{f} \mathbb{P}^1_{\mathbb{Q}}$

Assume: (1) all fibres at pts $\in \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q})$ are split

(2) Abelian splitting.

(3) $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$

then $\exists\ m_0 \in \mathbb{P}^1(\mathbb{Q})$ X_{m_0} smooth

and $X_{m_0}(\mathbb{A}_\mathbb{Q}) \neq \emptyset$

$\uparrow$

Green-Tao-Ziegler.

Cor: If fibres satisfy HP + WA

$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \Rightarrow \overline{X(\mathbb{Q})}^{\,r\cdot r} = X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}X}$

Example: $N_{K/\mathbb{Q}}(x_1\omega_1 + \dots + x_n\omega_n) = \prod_{i=1}^{d}(t - e_i)$

Ham and $K/\mathbb{Q}$ cyclic

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_{\mathrm{vr}}} \neq \emptyset \implies \overline{X(\mathbb{Q})}^{\mathrm{rep}} = X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}X}$$

$\Downarrow$

$\implies$ $X(\mathbb{Q})$ is Zariski dense in X

It uses

- $\mathrm{Br}\,k \to \bigoplus \mathrm{Br}\,k_v \to \mathbb{Q}/\mathbb{Z}$
- reciprocity argument à la Hilbert
- Harari's formal lemma

Conj:

$$\begin{array}{c} X \\ \big\downarrow f \\ \mathbb{P}^1_k \end{array}$$

fibration.

Assume that $X_\eta / k(\mathbb{P}^1)$ is a geometrically rationally connected variety

Assume that for almost all $m \in \mathbb{P}^1(k)$

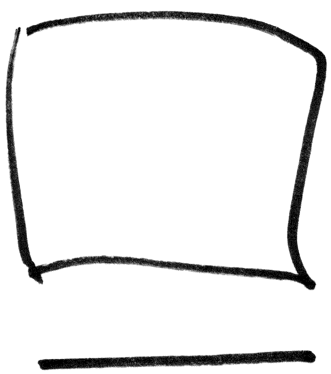
$$\overline{X_m(k)}^{top} = X_m(\mathbb{A}_k)^{\mathrm{Br} X_m}$$

then $\overline{X(k)}^{top} = X(\mathbb{A}_k)^{\mathrm{Br} X}$

ex $\quad N_{K/k}(\text{—}) = P(t)$

Harari: 1994 | 1997

Yes under condition $\delta = 1$.

X  X_η g. rat. connected

$\downarrow$

$\mathbb{P}^1_k$ $k(t)$

FACT Thm (Harari) $\exists$ Hilbert set $\subset \mathbb{P}^1(k)$

of points m such that

$$Br X_\eta / Br\, k(\mathbb{P}^1) \twoheadrightarrow Br X_m / Br\, k$$

$$H^1(k(t), Pic X_{\overline{k(t)}}) \to H^1(k_m, Pic \bar{X}_m)$$

Cor: $\left[\begin{array}{l} X \subset \mathbb{P}^4_k \quad \text{cubic hypersurface} \\ X \supset \mathbb{P}^1_k \\ \Rightarrow \quad \overline{X(k)}^{top} = X(\mathbb{A}_k) \end{array} \right]$

$$\begin{array}{ccc} X \leftarrow & X' & \\ & \downarrow & \\ & \mathbb{P}^2 & \end{array} \qquad \begin{array}{c} X'_m \\ | \\ m \end{array}$$

$\mathrm{Br}\, X' = \mathrm{Br}\, k$

$\mathrm{Br}\, X'_m / \mathrm{Br}\, k$ need not be zero.

2014

Harpaz–Wittenberg > Wittenberg 2013

get rid of the abelian splitting condition

~~Replace~~ Instead of $0 \to \mathrm{Br}\, k \to \bigoplus \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z} \to 0$

HW use Poitou–Tate duality for tori

T/k $\quad T \times_k \bar{k} \simeq \mathbb{G}_{m,\bar{k}}^n$.

$$H^2(k,T) \to \bigoplus_{v \in \Omega} H^2(k_v,T) \to \mathrm{Hom}(H^2(k,\hat{T}), \mathbb{Q}/\mathbb{Z})$$

lattice $\hat{T} = \mathrm{Hom}_{\bar{k}\text{-gr}}(\bar{T}, \mathbb{G}_m)$ $\qquad$ [illegible]

$T \times \hat{T} \to \mathbb{G}_m$

$T = R^1_{K/k} G_m$ $\qquad$ $\mathrm{Norm}_{K/k}(x_1\omega_1 + \dots + x_n\omega_n) = 1$

$H^1(k, T) = k^\times / \mathrm{Nm}\, K^\times$

$$\begin{tikzcd}
k^\times/NK^\times \arrow[r] & \bigoplus_v k_v^\times/NK_v^\times \arrow[r] \arrow[d, hook] & \mathrm{Hom}(H^1(k,\hat{T}), \mathbb{Q}/\mathbb{Z}) \arrow[d, hook] \\
\mathrm{Br}\, k \arrow[r] & \bigoplus \mathrm{Br}\, k_v \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

If $\underline{K/k \text{ cyclic}}$, $\mathrm{Br}\,k$ receives a hook arrow from $k^\times/NK^\times$.

$H^1(k, T) = H^1(\mathfrak{g}, T(\bar{k}))$

U/k variety

$H^1_{et}(U, T)$ classifies

torsors = principal homogeneous spaces PHS

torsors over U under T

$$\begin{array}{c} Y \\ {\scriptstyle T}\Big\downarrow \\ U \end{array}$$

$T \times Y \to Y$

acts faithfully and transitively

on the geometric fibres

Replace Harari's formal lemma for the Brauer gp by a ~~formallem~~ formal lemma for torsors under a torus

Prop: given U/k, $Y \to U$ a torsor under T.

Consider $\underline{H}^2(k, \hat{T}) \xrightarrow{\varphi} \mathrm{Br}\, U$

$\cdot$ fak $\quad u\,[Y \to U)$

$$H^2(k, \hat{T}) \times H^1(U, T) \to H^2(U, \mathbb{G}_m) = \mathrm{Br}\, U$$

$\underline{B = \mathrm{Im}\, \varphi}$ Assume that

$U \subset X$

smooth compact

Assume that

$$U(\mathbb{A}_k)^{B \cap Br X} \neq \emptyset$$

$(M_v) \in U(\mathbb{A}_k)$ $\quad \forall \alpha \in B \cap Br X$

$$\sum \alpha(M_v) = 0$$

S finite set of places

then $\exists\ \rho \in H^1(k, T)$ such that

for the twisted torsor $Y^\rho \longrightarrow U$

$\exists\ (N_v) \in Y^\rho(\mathbb{A}_k)$ $\quad N_v \longrightarrow M_v$ for $v \in S'$

Exempl

$\cup$ $$N_{K/k}(x_1\omega_1 + \dots + x_n\omega_n) = \prod_{i=1}^{d}(t-e_i) \neq 0 \quad *$$

$$F = \left(R^1_{K/k} G_m\right)^d$$

$$t - e_i = N_{K/k}(x_1^i\omega_1 + \dots + x_n^i\omega_n)$$

$i = 1 - d$

$\exists$ ... Ha $\exists$ $c_1, \dots, c_d \in k^*$

not k_v $\quad$ $0 \neq t - e_i = c_i N_{K/k}(\text{—})$ $\quad$ $i = 1 - d$

$\prod e_i = 1$

$$N(\ ) = \prod (t - e_i)$$

k = # field

Harpaz-Wittenberg   new hypothesis $e_i \in k$

(HW)   (*) $t - e_i = c_i \, \mathrm{Norm}_{L_i/k}(\Xi_i)$ $i = 1, \ldots, d.$

define the obvious finite bad set $S_0 \subset \Omega$

take any finite set $S \supset S_0$.

Assume given local solutions of (*) for $v \in S$

$\varepsilon > 0$

$\Longrightarrow$ (conj)

$\exists\, t_0 \in k$ s.t. $|t_0 - t_v|_v < \varepsilon \ \forall v \in S$

and for $v \notin S$

if $v(t_0 - e_i) > 0$

then $\exists\, w$ of L_i which is of degree 1 over v

/Q: Theorem of Matthiesen

Theorem $^{\times\text{H-W}}$ Modulo $(\overset{\text{hyp.}}{\text{HW}})$

$X \to \mathbb{P}^1_k$ fibration in rat. conn. varieties

$X(\mathbb{A}_k)^{Br} \neq \emptyset$

$\Longrightarrow \exists\, t_0 \in \mathbb{P}^1_k$ with X_{t_0} smooth

and $\left(X_{t_0}(\mathbb{A}_k)\right)^{Br\,X} \neq \emptyset$

$\to$ If $k = \mathbb{Q}$ unconditional theorem