

Cases when H^1 and WA hold.

• Quadratics.

• X/R smooth proj. g.c. $\exists G/R$ linear alg gp
 such
 $G \times X \rightarrow X \quad g_1(g_2 x) = (g_1 g_2)x \quad \forall x \in X(\bar{R})$
 transitive $\forall x_0 \in X(\bar{R}) \quad G(\bar{R})x_0 = X(\bar{R})$

• H^1 and WA for principal homogeneous

spaces of semi-simpl simply connected

linear alg gps

Eichler D/R

$$\alpha \in k^\times \cap \text{Nrd } D^\times \quad \frac{H^1(k, \text{SL}_n)}{\sim}$$

$$\iff \alpha \in k_v^\times \cap \text{Nrd } D_v^\times \quad \forall v \in \mathbb{Z}$$

There are many counterexamples to H^1 and WA

• Curves of genus > 0

• $\prod_{K/\mathbb{Q}} (x_1 w_1 + \dots + x_n w_n) = c$

e.g. K/\mathbb{Q} Galois

$$\text{Gal}(K/\mathbb{Q}) = \mathbb{Z}/2 \times \mathbb{Z}/2$$

K
|
 \mathbb{Q}

• PHS of G semisimple (not only unctd) (Serre)
for which H^1 fails

• $y^2 + z^2 = (x^2 - 2)(x^2 - 2)/\mathbb{Q}$

Iskrskikh

X/k smooth proj d.c.

$$X(k)^{\text{top}} \subset X(\mathbb{A}_k)^{\text{Br}\lambda} \subset X(\mathbb{A}_k) \quad 1970$$

Naire pyk.

Is $X(k)^{\text{top}} = X(\mathbb{A}_k)^{\text{Br}\lambda}$?

$$\downarrow$$
$$\pi X(k)$$

No (Shorohajtar)

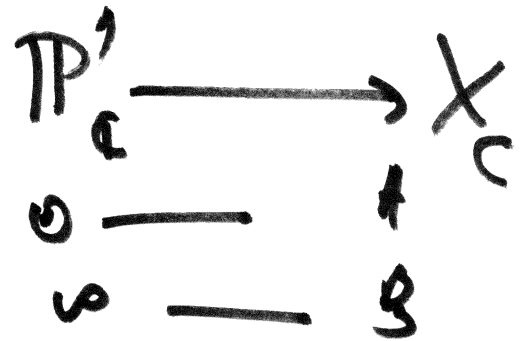
CONT [If X/k \bar{X} is RATIONALLY CONNECTED
then $X(k)^{\text{top}} = X(\mathbb{A}_k)^{\text{Br}\lambda}$]

Kollar-Miyaoka-Mori:

$$k \subset \bar{k} \subset \mathbb{C}$$

\bar{X} is rat connected ψ

any two pts $A, B \in X(\mathbb{C})$
are connected by a $\mathbb{P}^1_{\bar{k}}$



Known case of $X(k)^{top} = X(A_k)^{Br}$

Sakshe, Birrovi

⊗ $X \supset U$
smooth proj.

U is a homogeneous space
of a connected linear alg g, G

with $\forall x \in U(k)$

$G_x = \text{Stab}(x)$ is connected

Thm $X(k)^{top} = X(A_k)^{Br}$

Châtelet surfaces

$$y^2 - az^2 = P(z) \quad \text{separable} \quad (\text{X})$$

$$\deg P = 3 \text{ or } 4$$

X smooth curve

Thm $\left(\begin{array}{l} \text{X}, \text{ Gauss, Swinnerton-Dyer (1984-87)} \\ X(\mathbb{Q})^{\text{tr}} = X(\mathbb{Q}_p)/\text{Br} X \end{array} \right)$

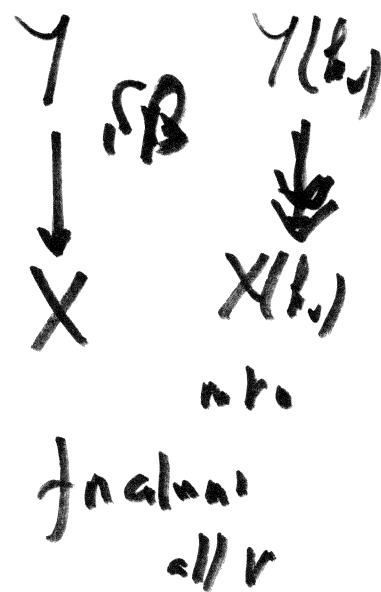
FACT $\left[\begin{array}{l} \text{this } X \text{ is not stably } h\text{-birational} \\ \text{to one of the previous ones} \end{array} \right.$

For $(*) \quad \text{Br } X_{\mathbb{Q}_r} / \text{Br } \mathbb{Q}_r = 0 \quad \text{for almost all } r.$

For $(**) \quad \text{not true}$

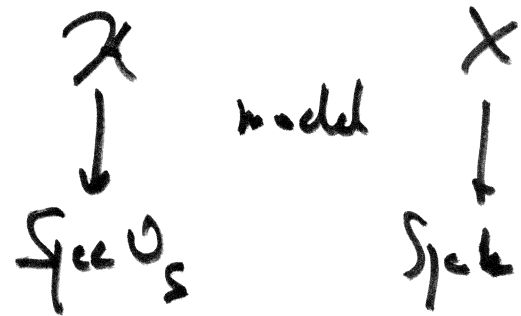
Pf using | descent
+ fibration

X/k # field smooth proj $A \in \mathbb{R}^n X$
 For all $v \in \Omega_k$ except finitely many
 $X(k_v) \xrightarrow{ev_A} \mathbb{B}^n(k_v)$
 has image $\neq \emptyset$



What if X is not projective?

Thm: X/k smooth g.c.



$U \subset X$
 φ_U
 $\alpha \in \text{Br } U \setminus \text{Br } X.$
 then \exists infinetly many $v \in \mathbb{A}^1_k$ such that
 $\exists M_v \in \mathcal{U}(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ and $\alpha(M_v) \neq 0$

Example. $U = \{y^2 - az^2 = P(t)Q(t) \neq 0\}$ irreducible
 $a \notin k^{x2}$ of odd degree

Ann $\prod U(k_v) \neq \emptyset$
 all v .

$S =$ obvious, local at
 of primes

$\alpha = (a, P(t))$ If $v \notin S'$ $a \notin k_v^{x2}$ (Tchebotarjev)

if I take $t_v = \frac{z}{\pi_v}$ $v(P(t_v)Q(t_v))$ even
 \Rightarrow solution $(y_v, z_v, t_v) = M_v \in U(k_v)$

If $v \notin S'$ $\exists M'_v \in U(k_v)$ $\alpha(M'_v) = 0$

$\Rightarrow \exists (M_v)_{v \in \Omega}$ such that $\sum_{v \in \Omega} \alpha(M_v) = 0$

~~$U(k_v)$~~
 $U(\mathbb{Q}_p)$

$$0 \rightarrow K^*/NK^* \rightarrow \bigoplus K_v^*/NK_v^* \rightarrow \mathbb{Z}/2 \rightarrow 0$$

$$K = k(\sqrt{a}) \quad 0 \rightarrow \text{Br } k \rightarrow \bigoplus \text{Br } k_v \rightarrow \bigcup \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

CFT
2

$$c \in k^* \rightarrow (P(t_r)) \rightarrow 0$$

(*)
(*)

$$\rightarrow \begin{cases} y_1^2 - a z_1^2 = c P(t) \neq 0 \\ y_2^2 - a z_2^2 = c^{-1} Q(t) \neq 0 \end{cases} \quad \left| \begin{array}{l} \text{has solutions} \\ \text{in all } k_v \end{array} \right.$$

Schwarz
implies
HP

(*)

$$\downarrow \\ y^2 - a z^2 = P(t)/Q(t)$$

if (*) has pts in all k_v , then $\exists c \in k^*$

(*)_c has pts in all k_v

Harari's formal lemma (1994)

X/k smooth
irreducible

$U \subset X$
open

$B \subset Br U$
finite subgroup

$(I_v) \in U(\mathbb{A}_k)$ ~~Assume:~~

Assume: $\forall \alpha \in B_n(Br X)$ $\sum_{v \in S} \alpha(I_v) = 0$

fix S' finite set.

then $\exists (I'_v) \in U(\mathbb{A}_k)$ with ~~$I'_v = I_v$~~
 $I'_v = I_v \quad v \in S$

such that

$\forall \alpha \in B \quad \sum_{v \in S'} \alpha(I'_v) = 0$

$$\underline{\text{Cor}} \quad y^2 - az^2 = \prod_{i=1}^h P_i(t) \neq 0 \quad P_i \text{ irreducible}$$

$$X \supset \emptyset$$

X smooth curve: If $X(\mathbb{A}_k) \neq \emptyset$

and if Schur's is true, then $X(\mathbb{R}) \neq \emptyset$.

$$(a, P_i(t)) \quad \left\{ P_i(t) = c_i (y^2 - az_i^2) \mid i=1-h \right\}$$

2010 Green, Tao, Ziegler.

$$l_i(u, v) = a_i u + b_i v \quad a_i, b_i \in \mathbb{Z}$$

$i = 1, \dots, n$

~~Assume~~ Assume some obvious restrictions —

then \exists extly many pairs of integers

~~(a, b)~~ (c, d) such that

$$\left\{ \begin{array}{l} l_2(c, d) = \text{prime} \\ \vdots \\ l_n(c, d) = \text{prime} \end{array} \right.$$

Browning - Matthiesse - Skorobogatov

Harper - Skorobogatov - Wittmers

Thm. $y^2 - az^2 = b \prod_{i=1}^{2n} (t - e_i) \neq 0$

$a \in \mathbb{Q}$
 $e_i \in \mathbb{Q}$
 $e_i \neq e_j$

\mathbb{Q}/\mathbb{Q}

$\mathbb{Q} \subset X$ smooth curve

Then

$$X(\mathbb{Q})^{\text{top}} = X(\mathbb{R}_{e_2})^{\text{Br}X}$$

Pf. $\nabla \quad y^2 - az^2 = b \prod_{i=1}^{2n} (u - e_i v) \neq 0 \quad (y, z, u, v)$

$y = \frac{Y}{v^n} \quad z = \frac{Z}{v^n} \quad t = \frac{u}{v} \quad \nabla \cong U \times \mathbb{G}_m$

(then) look at $B \subset \mathbb{P}^1 \times \mathbb{P}^1$
 defined by the quad eq. $(a, u - e_i v)$

Apply Hurwitz's formal lemma.

$c_i \in k^*$

has ord. w at all h_i

$$\left[\overline{y_i^2 - az_i^2 = c_i \cdot (u - e_i v)} \right]_{i=1-2n}$$

$b = \prod c_i$

$y^2 - az^2 = b \prod_{i=1}^{2n} (u - e_i v)$

u_0, v_0

$$\left[\begin{array}{l} y_n^2 - a z_n^2 = c_i \cdot \left[\overset{\in \mathbb{Q}}{y_0 - e_i v_0} \right] \neq 0 \\ i=1 \rightarrow l_n \end{array} \right]$$

has sol. in all $\mathbb{Q}_p = \mathbb{R}$

$$\text{and } c_i (y_0 - e_i v_0) = p_i \prod_{s \in I} \left(\right)^{v_s}$$