Cases when HP and WA hold.

- Quadrics.

- X/k smooth proj. g.c. $\exists\, G/k$ linear alg gp connected
  $G \times X \to X$ $\quad g_1(g_2 x) = (g_1 g_2)x \quad \forall x \in X(\bar{k})$
  transitive $\forall x_0 \in X(\bar{k}) \quad G(\bar{k})x_0 = X(\bar{k})$

- HP and WA for principal homogeneous spaces of semisimple simply connected linear alg gps

Eichler D/k $\qquad \alpha \in k^\times \cap \operatorname{Nrd} D^\times \quad \underline{H^1(k, SL(1))}$

$\Longleftrightarrow \alpha \in k_v^\times \cap \operatorname{Nrd} D_v^\times \ \forall v \in \Omega$

there are many counterexamples to HP $\overset{\text{or}}{\text{and}}$ WA

$$\begin{array}{c} K \\ | \\ \mathbb{Q} \end{array}$$

- Curves of genus > 0
- $N_{K/\mathbb{Q}}(x_1 w_1 + \dots + x_n w_n) = c$

  e.g. $K/\mathbb{Q}$ galois
  $\mathrm{Gal}(K/\mathbb{Q}) = \mathbb{Z}/2 \times \mathbb{Z}/2$
- PHS of G semisimple (not simply connected) (Serre)
  for which HP fails
- $y^2 + z^2 = (3 - x^2)(x^2 - 2) \,/\, \mathbb{Q}$
  Iskovskikh

X/k smooth proj g.c.

$$X(k)^{top} \subset X(\mathbb{A}_k)^{Br X} \subset X(\mathbb{A}_k) \qquad 1970$$

$$\prod X(k_v)$$

<u>Naive question</u>

Is $X(k)^{top} = X(\mathbb{A}_k)^{Br X}$?

<u>NO</u> (Skorobogatov)

<u>CONJ</u> If X/k $\bar{X}$ is RATIONALLY CONNECTED

then $X(k)^{top} = X(\mathbb{A}_k)^{Br X}$

Kollár–Miyaoka–Mori: $\qquad k \subset \bar{k} \subset \mathbb{C}$

$\bar{X}$ is rat connected if

any two pts $A, B \in X(\mathbb{C})$

are connected by a $\mathbb{P}^1_{\mathbb{C}}$

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & X_{\mathbb{C}} \\ 0 & \text{—} & A \\ \infty & \text{—} & B \end{array}$$

Known cases of $X(k)^{top} = X(\mathbb{A}_k)^{Br}$

Sansuc, Borovoi

$(\forall)$ $X \supset U$ smooth proj

U is a homogeneous space of a connected linear alg gr G

with $\forall x \in U(\bar{k})$

$G_x = \mathrm{Stab}(x)$ is connected

<u>Thm</u> $\underline{X(k)^{top} =} X(\mathbb{A}_k)^{Br}$

Châtelet surfaces

$$y^2 - az^2 = P(x) \quad \text{separable} \quad (**)$$

$$d^0 P = 3 \text{ or } 4$$

X smooth compact.

Thm (C-T, Sansuc, Swinnerton-Dyer 1984-87)

$$X(k)^{top} = X(\mathbb{A}_k)^{Br X}$$

FACT: this X is not stably k-birational to one of the previous ones

For $(*)$ $\quad Br X_{k_v} / Br k_v = 0$ for almost all v.

For $(**)$ $\quad$ not true

Pf using | descent + fibration

X/k smooth proj $\#$ field $\quad A \in \mathrm{Br}\,X$

for all $v \in \Omega_k$ except finitely many

$$X(k_v) \xrightarrow{ev_A} \mathrm{Br}\,k_v$$

has image $= 0$

$$\begin{array}{ccc} Y & & Y(k_v) \\ \downarrow S_B & & \downarrow \\ X & & X(k_v) \end{array}$$

onto for almost all v

What if X is not projective?

Thm: X/k smooth g.c.

$$\begin{array}{ccc} \mathcal{X} & \text{model} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_S & & \operatorname{Spec} k \end{array}$$

$U \subset X$ open, $\alpha \in \operatorname{Br} U \setminus \operatorname{Br} X$.

then $\exists$ infinitely many $v \in \Omega_k$ such that

$\exists\ M_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ and $\alpha(M_v) \neq 0$

<u>Example.</u> $U = \boxed{y^2 - a z^2 = P(t)\,Q(t) \neq 0}$ irreducible of <u>odd</u> degree

$a \notin k^{\times 2}$

Assume $\prod_{\text{all } v} U(k_v) \neq \emptyset$

$\boxed{S = \text{obvious bad set of primes}}$

$\alpha = (a, P(t))$ If $v \notin S'$ $a \notin k_v^{\times 2}$ (Tchebotarev)

if I take $t_v = \dfrac{1}{\pi_v}$ $v(P(t_v)\,Q(t_v))$ even

$\exists$ solution $(y_v, z_v, t_v) = M_v \in U(k_v)$

If $v \notin S'$ $\exists\, M'_v \in U(k_v)$ $\alpha(M'_v) = 0$

$\Longrightarrow \exists\ (M_v)_{v \in \Omega}$ such that $\sum_{v \in \Omega} \alpha(M_v) = 0$

$\cap$

~~$\prod U(k_v)$~~

$U(\mathbb{A}_k)$

$$\begin{tikzcd}
0 \arrow[r] & k^*/NK^* \arrow[r] & \bigoplus k_v^*/NK_v^* \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \\
K = k(\sqrt{a}) \quad 0 \arrow[r] & \mathrm{Br}\,k \arrow[r] \arrow[u, phantom, "\cap"] & \bigoplus \mathrm{Br}\,k_v \arrow[r] \arrow[u, phantom, "\cap"] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[u, phantom, "\cap"] & 0 \\
& c \arrow[r] & (P(t_v)) \arrow[r] \arrow[uu] & 0 &
\end{tikzcd}$$

$\underline{\underline{\text{CFT}}}$

$\exists$ $\quad \in k^*$

$$\longrightarrow \quad \begin{cases} y_1^2 - a z_1^2 = c\, P(t) \neq 0 \\ y_2^2 - a z_2^2 = c^{-1} Q(t) \neq 0 \end{cases}$$

$\boxed{**}$ $\quad \exists c \in k^*$

has solution in all k_v

Schinzel implies HP

$$\downarrow$$

$(*)$ $\quad y^2 - a z^2 = P(t)\,Q(t)$

If $(*)$ has pts in all k_v, then $\exists c \in k^*$

$\binom{*}{*}_c$ has pts in all k_v

Harari's formal lemma (1994)

X/k smooth irreducible $\qquad U \subset X$ open $\qquad B \subset \mathrm{Br}\, U$ finite subgp

$(P_v) \in U(\mathbb{A}_k)$ ~~Assume~~ :

Assume: $\forall\, \alpha \in B \cap (\mathrm{Br}\, X) \quad \sum_{v \in \Omega_k} \alpha(P_v) = 0$

fix S finite set.

Then $\exists\, (P'_v) \in U(\mathbb{A}_k)$ with $\cancel{P'_v = P_v}$ $P'_v = P_v \; v \in S$

such that
$$\forall\, \alpha \in B \qquad \sum_{v \in \Omega} \alpha(P'_v) = 0$$

Cor $y^2 - a z^2 = \prod_{i=1}^{n} P_i(t) \neq 0$ P_i irreducible

$X \supset U$

X smooth compact. If $X(\mathbb{A}_k)^{\mathrm{Br}\,X} \neq \emptyset$

and if Schinzel is true, then $X(k) \neq \emptyset$.

$(a, P_i(t))$ $\left\{ P_i(t) = c_i (y_i^2 - a z_i^2) \;\middle|\; i = 1 - n \right\}$

2010 Green, Tao, Ziegler.

$$l_i(u,v) = a_i u + b_i v \qquad a_i, b_i \in \mathbb{Z}$$

$i = 1, \dots, n$

Assume some obvious restrictions —

then $\exists$ ∞ly many pairs of integers

(c, d) such that

$$\begin{cases} l_1(c,d) = \text{prime} \\ \quad \vdots \\ l_n(c,d) = \text{prime} \end{cases}$$

Browning – Matthiesen – Skorobogatov

Harpaz – Skorobogatov – Wittenberg

Thm. $y^2 - az^2 = b\prod_{i=1}^{2n}(t-e_i) \neq 0$ $\quad a \in \mathbb{Q}$, $e_i \in \mathbb{Q}$, $e_i \neq e_j$

$U/\mathbb{Q}$ $\quad U \subset X$ smooth compact

then $\boxed{X(\mathbb{Q})^{top} = X(\mathbb{A}_\mathbb{Q})^{Br X}}$

Pf. $V \quad Y^2 - aZ^2 = b \prod_{i=1}^{2n} (u - e_i v) \neq 0 \quad (Y, Z, u, v)$

$$y = \frac{Y}{v^n} \qquad z = \frac{Z}{v^n} \qquad t = \frac{u}{v} \qquad V \cong U \times \underset{t}{\mathbb{G}_m}$$

then look at $B \subset \mathrm{Br}\, V$

spanned by the quat. alg. $(a, u - e_i v)$

Apply Harari's formal lemma.

$\rightsquigarrow \quad c_i \in k^* \qquad \leftarrow \quad \boxed{Y_i^2 - aZ_i^2 = c_i \cdot (u - e_i v)} \quad i = 1 - 2n$

has sol. in all k_v

$\downarrow$

$b = \prod c_i \qquad Y^2 - aZ^2 = b \prod_{i=1}^{2n} (u - e_i v)$

u_0, v_0

$$\left[Y_1^2 - a Z_1^2 = c_i \left(u_0 - e_i v_0 \right) \neq 0 \quad i = 1 \ldots l_1 \right)$$

has sol. in all $\mathbb{Q}_p$ & $\mathbb{R}$

and $c_i(u_0 - e_i v_0) = p_i \prod_{s \in S} (q_s)^{\mu_s}$