X/k smooth proj. geom. irreducible

$\bar{X} = X \times_k \bar{k}$

$\bar{k}$ sep. closure of k.

$$\begin{tikzcd}
\mathrm{Br}\,k \arrow[r] & \underbrace{\mathrm{Ker}[\mathrm{Br}\,X \to \mathrm{Br}\,\bar{X})}_{\mathrm{Br}_a X} \arrow[r] & H^2(k, \mathrm{Pic}\,\bar{X}) \arrow[d] \\
 & & H^3(k, \mathbb{G}_m)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0\bar{X} \arrow[r] & \mathrm{Pic}\,\bar{X} \arrow[r] & NS(\bar{X}) \arrow[r] & 0 \\
 & \text{abelian variety} \arrow[u] & & &
\end{tikzcd}$$

Assume $NS(\bar{X})_{\mathrm{tors}} = 0$

[Assume $\mathrm{Alb}_{\bar{X}} = 0$ then $H^2(k, \mathrm{Pic}\,\bar{X})$ is finite.

$\mathrm{Im}\left[\mathrm{Br}\,X \longrightarrow \mathrm{Br}\,\bar{X}^{G}\right]$ = "transcendental part of the Brauer gp

This is finite.

Basic conj. $\forall X/k$

k of finite type over $\mathbb{Q}$

If $\bar{X} \sim \mathbb{P}^n_{\bar{k}}$ $\mathrm{Br}\,\bar{X} = 0$

The fibration method

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

k number field.

X, Y smooth, proj. g.i.

X_η is geometrically integral

Ask: If Y and the fibres X_m, $m \in Y(k)$ satisfy HP + WA $\overset{?}{\Longrightarrow}$ X satisfies HP + WA

Mostly $Y = \mathbb{P}^1_k$

Conics / k. $\qquad C/k \qquad C \Longleftrightarrow (a,b)$

$$\prod_{v\in\Omega} C(k_v) \neq \emptyset \Longrightarrow C(k) \neq \emptyset .$$

$$\prod_{v\in\Omega\setminus\{v_0\}} C(k_v) \neq \emptyset \Longrightarrow C(k_{v_0}) \neq \emptyset$$

and $C(k) \neq \emptyset$

$$0 \to \operatorname{Br} k \to \bigoplus \operatorname{Br} k_v \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

cplex

$$(a,b) \longrightarrow (0 \text{---} 0, \tfrac{1}{2}, 0 \text{--}) \longrightarrow 0$$

$$\operatorname{Br} k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

HP (Main) for ~~$ax^2+by^2-cz^2-dt^2=0$~~

$$ax^2+by^2-cu^2-dv^2=0$$

$$ax^2+by^2=t=cu^2+dv^2\neq 0$$

$S = \{ v \quad v(a)\neq 0 \text{ a } v(b)\neq 0 \quad v(2)\neq 0 \}$

bad set $\quad$ v archimedean.

$v \in S'$ $\quad ax_v^2+by_v^2 = \underset{\in k_v^*}{t_v} = cu_v^2+dv_v^2 \neq 0$

$\varepsilon > 0$

there exists $\quad t_0 \in k^* \quad |t_0-t_v|_v < \varepsilon \quad v \in S'_{fin}$

such that $\quad (t_0) = \mathfrak{g} \underline{\prod \mathfrak{q}^{n_{\mathfrak{q}}}} \quad n_{\mathfrak{q}} \in \mathbb{Z}$

$\mathfrak{q}$ above S

$\uparrow$

one prime

$$\underbrace{ax^2+by^2}=t_0=cu^2+dv^2$$

$\boxed{ax^2+by^2=t_0}$ Conic has pts in k_v $v\in S_\infty$

in k_v $v\notin S\cup(\mathfrak{g})$

$a\in O_v^*$

$b\in O_v^*$

$t_0\in O_v^*$

use Hensel Conic has pts in k_v $\underline{v\in S'}$

provided ε small enough.

$ax_v^2+by_v^2=t_v\neq 0$ implicit function theorem

has pts in all k_v except possibly $k_{\mathfrak{g}}$.

$\Longrightarrow$ $ax_0^2+by_0^2=t_0$ | $t_0=cu_0^2+dv_0^2$

reciprocity +

$\exists$ $\in k$

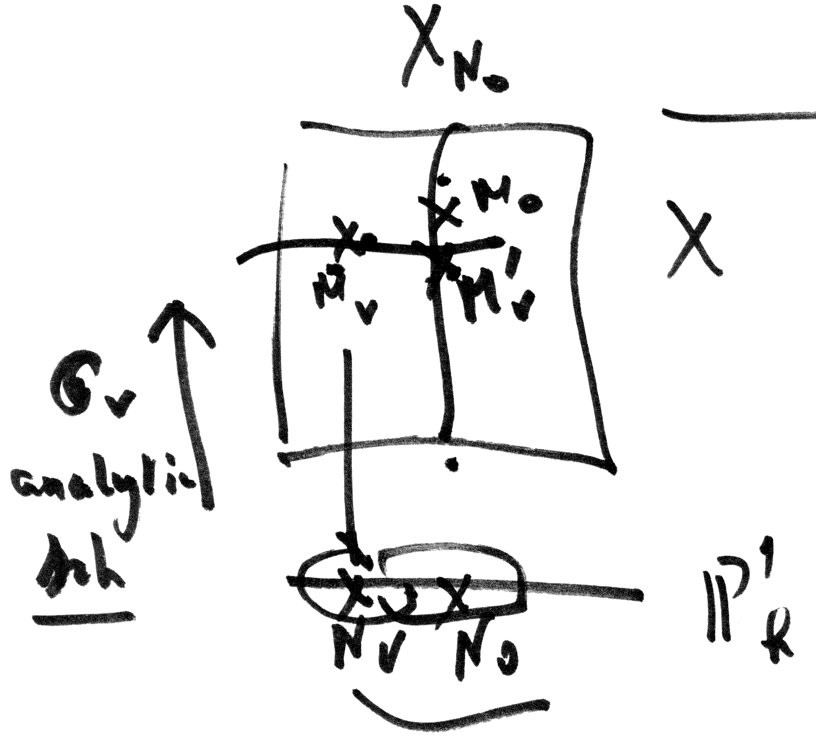
$\exists\, e$

Prop. $\boxed{ax^2+by^2+cz^2 = dt^2+e \neq 0}$

satisfies HP + WA.

Pf. Only uses

- implicit function theorem
- weak approx on $\mathbb{P}^1_k$



$\ldots X_{N_0}(k) \neq 0$

How to get also WA $\leftarrow$

Schinzel's hypothesis (H)

$P_i(t) \in \mathbb{Z}[t]$ irreducible, leading coeff > 0

$i = 1, \dots, n$

Assume: ~~$\prod_{n \in \mathbb{Z}} P_i(n)$~~ $\left(P_i(n)\right) = 1$

then: $\left[\begin{array}{l} \exists \text{ infinitely many } n \in \mathbb{N} \\ \text{s.t. each } P_i(n) \text{ is a prime} \end{array}\right]$

---

ex

Examples $at+b$ Dirichlet.

$t, t+2$ twin primes

t^2+1

Z t^2+t+2

Z $t, t+2, t+4$

(H^*) k number field

$P_i(t) \in k[t]$ irreducible. monic.

S finite set of places $S \supset v | p \leq \sum d^\circ P_i$

$t_v \in k_v$ $\varepsilon > 0$

$\xrightarrow{(H)}$ $\exists\, t_0 \in k$ $|t_0 - t_v|_v < \varepsilon$

(Serre) $t_0 >> 0$ at real places

~ 1990 each $P_i(t_0) = \mathfrak{p}_i \cdot \prod \mathfrak{q}^{n_{\mathfrak{q}}}$

\ above S

Thm*.

Sansuc + I

1979

If (H) is true, then equations $y^2 - az^2 = P(t) \neq 0$ $\quad U/k \quad a \in k^*$

↑ irreducible poly.

satisfy HI and WA

Pf | Mimick Hasse's pf of HI using Dirichlet

FACT | $U \subset X$ smooth compact

$\underline{\operatorname{Br} X / \operatorname{Br} k = 0}$ because I is irreducible

$q(x,y) = t = q'(u,v)$ ②

$q(x,y,z) = P(t)$ ①

$y^2 - az^2 = P(t)$

$$\begin{tikzcd} X \arrow[d, "f"] \\ \mathbb{P}^1_k \end{tikzcd}$$

① $\nexists f$ s.t all the fibers X_m $m \in \mathbb{P}^1_k$ are geometrically integral

$\delta = 0$

② Bad fibers $t=0$ $t=\infty$.

$q(x,y)=0$

$q'(u,v)=0$

$\delta = 2$

③ Bad fibers m ∞

deg 1

$\delta = \deg P(t)$

$$\delta = \sum_{m \in \mathbb{P}^1_k} (k(m):k)$$

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{P}^1_k \end{tikzcd}$$

$\boxed{\delta \overset{?}{=} 0}$

Prop then $\exists\, S_0 \subset \Omega$ finite

s.t. $\forall\, v \notin S_0$

$X(k_v) \longrightarrow \mathbb{P}^1(k_v)$ onto

Pf use spreading out + Lang-Weil estimates ~1950

X/k geometrically int. of a given geometric type

$\exists\, S \subset \Omega_k \quad S = S(\text{geom}) \quad \forall\, v \notin S$

$X(k_v) \neq \emptyset$

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{P}^1_k \ni m \end{tikzcd}$$

X_m is split if $\exists\ Y \subset X_m$
$/k(m)$ component COMPONENT
which is of multiplicity one
and geometrically integral over $k(m)$,

Skorobogatov

Thm (Denef) X, Y smooth proj. geom. integral $/k$ (k field)

$f : X \longrightarrow Y$. dominant, generic fibre g.i.

$X_\eta \longrightarrow \eta$

Assume that for each $k \subset A \subset k(Y)$ dvr

$\exists$ $\mathcal{X}_A \xrightarrow[f_A]{} \operatorname{Spec} A$ flat proj. s.t. $\mathcal{X}_{k_A}$ is split

$\Longrightarrow$ $\exists$ S finite $\subset \Omega_k$ s.t.

$X(k_v) \longrightarrow Y(k_v)$ onto for $v \notin S$

Corollary. Combining Kollár on Ax's conj
with a trick

Deny $\Longrightarrow$ | "new" proof of Ax-Kochen theorem. |

$\mathbb{Q}_p$ $\quad d > 0 \quad n > d^2$

Then $\exists$ S finite $\subset \Omega_{\mathbb{Q}}$ s.t. $S = (d,n)$

for $p \notin S$ any form of degree d in | $/\mathbb{Q}_p$
n variables has a nontrivial zero over $\mathbb{Q}_p$.
