k $\quad k_s \quad$ $\mathcal{G} = \mathrm{Gal}(k_s/k)$

$k_s^\times \qquad H^1(\mathcal{G}, k_s^\times) = 0 \qquad$ Hilbert 90

$$H^2(\mathcal{G}, k_s^\times) \simeq \mathrm{Br}\, k$$

<u>Tsen</u> $\mathrm{Br}(k(C)) = 0$ $\quad k = k_{alg} \quad C/k$ curve

Lang

$$\begin{tikzcd} A \subset K \arrow[d, "\text{dvr}"'] \\ \mathbb{K} \end{tikzcd}$$

$\mathbb{K}$ alg closed $\quad A$ complete

$\mathrm{Br}\, K = 0$

$$1 \to \mu_n \to k_s^\times \longrightarrow k_s^\times \to 1 \qquad (n, \mathrm{char}\, k) = 1$$

$$x \longmapsto x^n$$

$$H^2(k, \mu_n) \simeq \mathrm{Br}\, k[n]$$

X scheme $\qquad \mathrm{Br}_{Az}(X)$

Grothendieck $\qquad \mathrm{Br}(X) := H^2_{et}(X, \mathbb{G}_m)$

étale topology, étale cohomology.

$$H^1_{et}(X, \mathbb{G}_m) = H^1_{Zar}(X, \mathbb{G}_m) = \mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{div}(f)$$

rat. funct.

$$1 \to \mu_n \to \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^n} \mathbb{G}_{m,X} \to 1 \qquad n \in \Gamma(X, \mathcal{O}_X^*)$$

$$0 \to \mathrm{Pic}X/n \to H^2_{et}(X, \mu_n) \to \mathrm{Br}X[n] \to 0$$

X regular integral scheme $\qquad k(X) =$ function field of X

$$0 \to \mathbb{G}_{m,X} \to i_{\eta *}\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X^{(1)}} i_{x*}\mathbb{Z} \to 0$$

$\eta =$ generic pt.

$$\begin{tikzcd}
& & A^\times \arrow[r] \arrow[d, "\oplus"] & K^\times \arrow[r] & \bigoplus \mathbb{Z} \arrow[d, phantom, "\wr"] \\
H^2\big(X, \bigoplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}\big) \arrow[r] \arrow[d, equal] & H^2_{\text{et}}(X, \mathbb{G}_m) \arrow[r] & H^2_{\text{et}}(X, i_{\eta *}\mathbb{G}_m) \arrow[r] \arrow[d] & \bigoplus_{x \in X^1} H^2(X, i_{x*}\mathbb{Z}) \arrow[d, equal] \\
\bigoplus H^2(k(x), \mathbb{Z}) \arrow[d, equal] & & H^2(\eta, \mathbb{G}_m) \arrow[d] \arrow[r, dotted] & \bigoplus H^2(k(x), \mathbb{Z}) \arrow[d, equal] \\
0 & & \vdots & \bigoplus H^1(k(x), \mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

Residues

Thm $\boxed{\mathrm{Br}\, X \hookrightarrow \mathrm{Br}\, k(X)}$

Purity for étale cohomology

$Y \subset X \supset U = X \setminus Y$ $\qquad$ $n \in \Gamma(X, \mathcal{O}_X^*)$

closed subscheme of pure codim c $\qquad$ X regular

Y regular

$$H^2(X, \mu_n) \to H^2(U, \mu_n) \to H^3_Y(X, \mu_n)$$

Want: $\quad H^i_Y(X, \mu_n) \cong H^{i-2c}(Y, \mu_n^{\otimes 1-c})$

1968 $\quad$ proved SGA 4 $\quad$ for X/k smooth variety / field

true for excellent schemes.

book in Astérisque $\quad$ Gabber $\quad$ Illusie, Orgogozo

$$\overset{\text{pure}}{Y} \subset X \supset U$$

$\amalg Y_i$ codim 1, irreducible

$$\operatorname{Pic} X \twoheadrightarrow \operatorname{Pic} U$$

$$\underline{H^2(X, \mu_n)} \to H^2(U, \mu_n) \to \bigoplus_{i=1}^{r} H^1(Y_i, \mathbb{Z}/n)$$

$$H^1(Y_i, \mathbb{Z}/n) \subset H^1(k(Y_i), \mathbb{Z}/n)$$

$$\operatorname{Br} X[n] \to \operatorname{Br} U[n] \to \bigoplus_{i=1}^{m} H^1(k(Y_i), \mathbb{Z}/n)$$

$$H^1(k(Y_i), \mathbb{Z}/n)$$

$$Y \subset X \supset U$$

$$Y_{sg} \subset X \setminus Y_{sg} \supset U \qquad \operatorname{Br}(X \setminus Y_{sg}) \to \operatorname{Br}(U) \xrightarrow{\ \partial\ } \bigoplus_{i=1}^{m} H^1(k(Y_i), \mathbb{Z}/n)$$

Thm. X regular, integral, excellent purity for Br) $\ell \in \mathcal{O}_X^*$

Brauer gr

$$0 \to \mathrm{Br}\,X\{\ell\} \to \mathrm{Br}\,k(X)\{\ell\} \to \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Q}/\mathbb{Z})\{\ell\}$$

A excellent dvr

$\ell \in k^\times$

$A \subset K$, $A \downarrow k$, char 0

$$\mathrm{Br}'K \to H^1(k, \mathbb{Q}/\mathbb{Z})$$

$$\begin{array}{ccc} \bar{\hat{K}} & & \\ \mid \, I & & \\ \hat{K}_{nr} & - & k_s \\ G \mid & & \mid \\ \hat{K} & - & k \end{array}$$

k perfect

$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$

$A \to \hat{A} \subset \hat{K}$

$$\mathrm{Br}(\hat{K}) = \mathrm{Br}(\hat{K}, \hat{K}_{nr}) = H^2(G, \hat{K}_{nr}^\times) \xrightarrow{\sim} H^2(G, \mathbb{Z})$$

$$\cong H^1(G, \mathbb{Q}/\mathbb{Z})$$

$$1 \to U^\times \to K_{nr}^* \xrightarrow{v} \mathbb{Z} \to 0$$

RES

Consequence of purity is birational invariance

$/k$ $\quad \operatorname{char} k = 0$ $\qquad X, Y / k$ smooth proj integral k-variety

$$X \supset U \simeq V \subset Y$$

Thm. then $\operatorname{Br} X \simeq \operatorname{Br} Y$.

Pf

$$\begin{tikzcd}
X \arrow[r, dashed] & Y \\
W \arrow[u, phantom, "\cup" description] \arrow[ur, "\varphi"'] & \\
X^{(1)} \arrow[u, phantom, "\cup" description] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Br} Y \arrow[r, phantom, "\subset"] \arrow[d] & \operatorname{Br} k(Y) \\
\operatorname{Br} W \arrow[r, phantom, "\subset"] & \operatorname{Br} k(X) \\
\operatorname{Br} X \arrow[u, "\cong"'] &
\end{tikzcd}$$

k perfect.

$$1 \to \bar{k}^* \to \bar{k}(t)^* \to \bigoplus_{x \in \mathbb{P}^{1\,(1)}_{\bar{k}}} \mathbb{Z} \to \mathbb{Z} \to 0$$

Use HSeq.:

$$0 \to \operatorname{Br} k \to \operatorname{Br} k(t) \to \bigoplus_{x \in \mathbb{P}^{1\,(1)}_{k}} H^1(k(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{cores}} H^1(k, \mathbb{Q}/\mathbb{Z}) \to 0$$

<u>Faddeev</u>

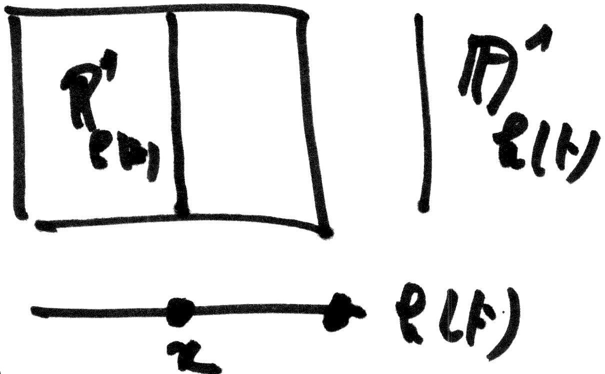
$\longrightarrow$ $\operatorname{Br} k = \operatorname{Br} \mathbb{P}^1_k$ $\longleftarrow$ true for any k

$\longrightarrow$ $\operatorname{Br}' k = \operatorname{Br}' \mathbb{A}^1_k$ $\longleftarrow$ $\mathbb{Z}$ for $p =$ char k.

char $k = 0$ $\rightsquigarrow$ $\operatorname{Br} k \xrightarrow{\sim} \operatorname{Br} \mathbb{A}^1_k$

Pf

$$\mathbb{A}^2 \longrightarrow \mathbb{A}^1$$



$x \in \mathbb{P}_k^{1\,(1)}$

$$\begin{tikzcd}
\overset{\beta}{\mathrm{Br}\,\mathbb{A}^2_k} \arrow[r] & \overset{\beta_1}{\mathrm{Br}\,\mathbb{A}^1_{k(t)}} \arrow[r, "\partial"] & H^1(\cancel{\ \ }\ k(x)(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}) \\
\mathrm{Br}\,k \arrow[u] \arrow[r] & \mathrm{Br}\,k(t) \arrow[u, "\cong"] \arrow[r, "\partial"] & H^1(k(x), \mathbb{Q}/\mathbb{Z}) \arrow[u, hook]
\end{tikzcd}$$

ω_j

$$\beta_1 \longrightarrow \beta_2 \longrightarrow 0 \quad ..$$

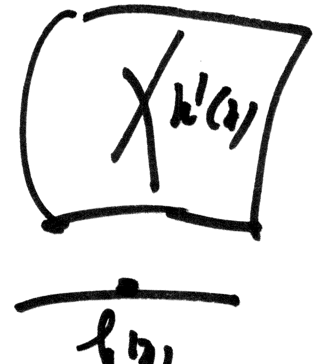
* <u>Conics</u>. C/k smooth conic $\overset{x^2 - ay^2 - bz^2 = 0}{\longleftrightarrow}$ (a,b)

$$0 \to \mathbb{Z}/2\,(a,b) \to \operatorname{Br} k \to \operatorname{Br} C \to 0$$

Witt 1934

$$\begin{array}{cc} X & X_\eta = \underline{\text{Conic}}. \\ \downarrow & | \\ \mathbb{P}^1_k & k(t) \end{array}$$

X/k smooth proj.



$X_\eta/k(t)$

$\updownarrow$

$\alpha \in \operatorname{Br} k(t)$

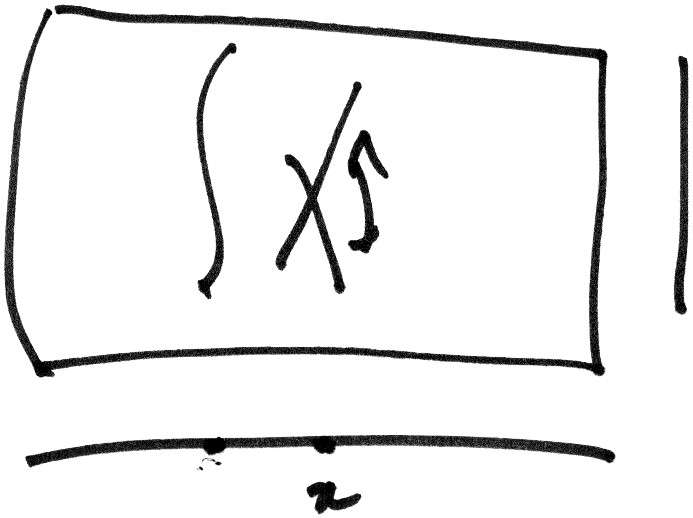
~~Assume~~ Assume that $X_\eta(k(t)) = \emptyset$.

Then exact sequence

$$\boxed{0 \to \operatorname{Br} k \to \operatorname{Br} X \to \Big[\bigoplus_{x \text{ bad}} \mathbb{Z}/2\Big] \Big/ \rho_x(A) \xrightarrow{\text{Cores}} \mathbb{Z}/2}$$

$$1 \to U_{nr}^{\times *} \to U_{nr}^{\times} \to K_0^{\times} \to 1$$

$$H^2(G, U_{nr}^{\times}) \to H^2(G, K_{nr}^{\times}) \to H^2(G, \mathbb{Z})$$



cnvi

$$\begin{tikzcd}
 & & 0 & & \\
\beta \arrow[r] & \beta_1 & & & \\
\mathrm{Br}\,X \arrow[r] & \mathrm{Br}\,X_\eta \arrow[u] \arrow[rr] & & \bigoplus_{\text{good } x} H^1(k(x), \mathbb{Q}/\mathbb{Z}) & \bigoplus_{\text{bad}} \\
 & \beta_2\ \mathrm{Br}\,k(t) \arrow[u] \arrow[u, shift left] \arrow[rr] & & \bigoplus_{x \in \mathbb{P}^1} H^1(k(x), \mathbb{Q}/\mathbb{Z}) \arrow[u] \arrow[ur] & \\
 & \mathbb{Z}/2(d) \arrow[u] & & & \\
 & \int & & &
\end{tikzcd}$$

at bad x

$H^1(k'(x)/k(x), \mathbb{Z}/4)$

$\shortparallel$

$\mathbb{Z}/2$