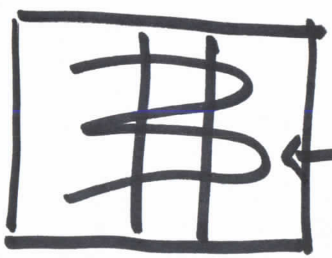
To count cubic rings we need to count $Gl_2(\mathbb{Z})$ classes of

$$ax^3+bx^2y+cxy^2+dy^3$$

$f = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_0 y^n \qquad a_i \in \mathbb{Z}$



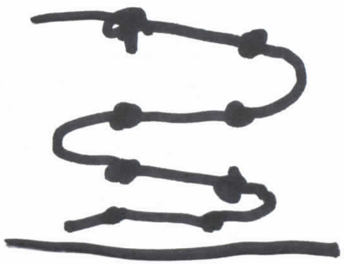
$\mathbb{P}^1_{\mathbb{Z}}$

V_f where $f=0$

Spec $\mathbb{Z}$

Global functions on V_f

$\underset{R_f}{\overset{\parallel}{H^0(V_f,\mathcal{O})}}$ rank n ring



Spec R_f

Spec $\mathbb{Z}$

$(a_1, \dots, a_n) = 1$

$\mathbb{P}^1_{\mathbb{Z}}$

$V_f = \operatorname{Spec} R_f$

$\operatorname{Spec} \mathbb{Z}$

rank n ring (e.g. maxl order in deg n # field) R

$\operatorname{Spec} R_f$

$\operatorname{Spec} \mathbb{Z}$

Spec R_f $\supseteq$ has a canonical embedding $\mathbb{CP}^N$

What is a map Spec $R_f \to \mathbb{P}^N$?

- line bundle on Spec R_f
- $N+1$ sections of bundle, nowhere all vanishing

element of class group of R_f

elements of the ideal

canonical embedding of
Spec R_f
given ideal class of inverse different

How many sections do you need so they are nowhere all vanishing?
In general need $n-1$.
cubic ring Spec $R_f \subset \mathbb{P}^1_{\mathbb{Z}}$

Rmk For every n, some $\operatorname{Spec} R_f$ deg n do embedd in $\mathbb{P}^1_{\mathbb{Z}}$. These are particularly nice rank n rings (e.g. orders in deg n # fields).

· Analogy to plane curves.

Thm $GL_2(\mathbb{Z}) \times GL_n(\mathbb{Z}) \times GL_n(\mathbb{Z})$ orbits of $\mathbb{Z}^2 \times \mathbb{Z}^n \times \mathbb{Z}^n$ parametrize class groups of these $\operatorname{Spec} R_f \subset \mathbb{P}^1_{\mathbb{Z}}$.

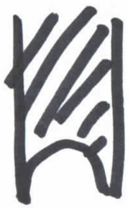
$n=3$ all cubic rings are this nice.

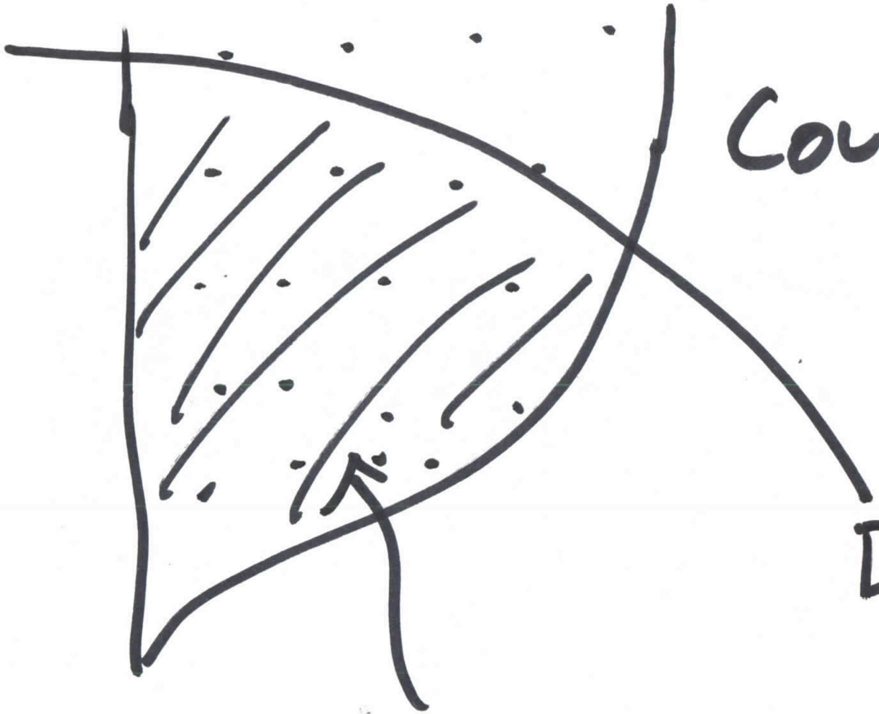
How to count $GL_2(\mathbb{Z})$ orbits of (a,b,c,d)?

$\mathbb{Z}^4$ 4 diml lattice of (a,b,c,d)

Need a fund. domain F for action of $GL_2(\mathbb{Z})$

each orbit has exactly 1 lattice point in F





Count lattice points in F

Geometry of numbers

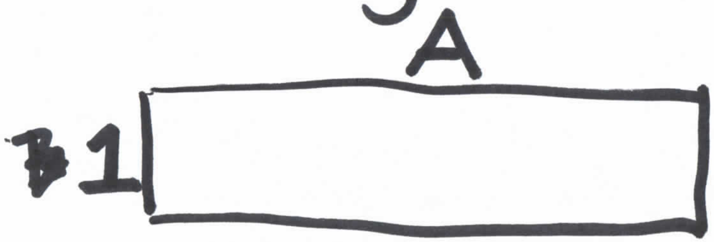
- First example: region scaling with X
  B ball # pts in $B \cdot X \approx \mathrm{vol}(B) X^{\dim}$

region R + I want to count lattice points in R ($N(R)$ = # lattice pts in R)

Hope:

$$N(R) \approx \mathrm{Vol}(R)$$

How big can $N(R) - \mathrm{Vol}(R)$ be?



Vol = A

could have as many as 2A points

as few as zero

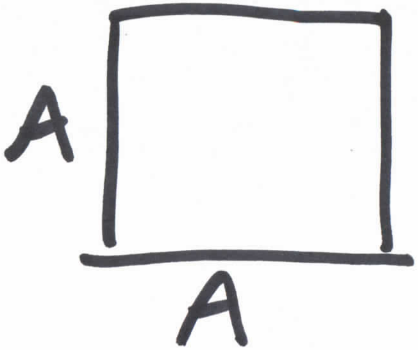
Thm (Davenport)

$$\left| N(R) - \mathrm{Vol}(R) \right| = O\left(\mathrm{Vol}(\mathrm{Proj}(R)) \right)$$

depends on dim, degree eqns defining bdries of R

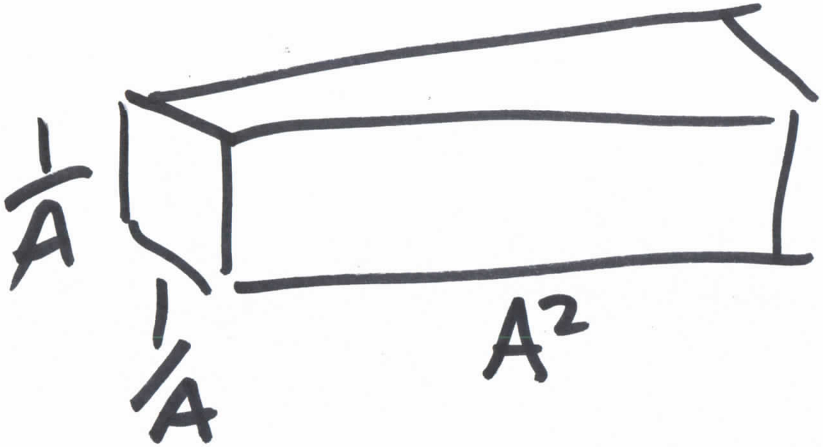
Vol(Proj(R))

projections of R onto all coordinate hyperplanes (of every lower dim)



$\mathrm{Vol} = A^2$

$\mathrm{Vol}(\mathrm{Proj}) = A$



Vol 1

2diml Projs

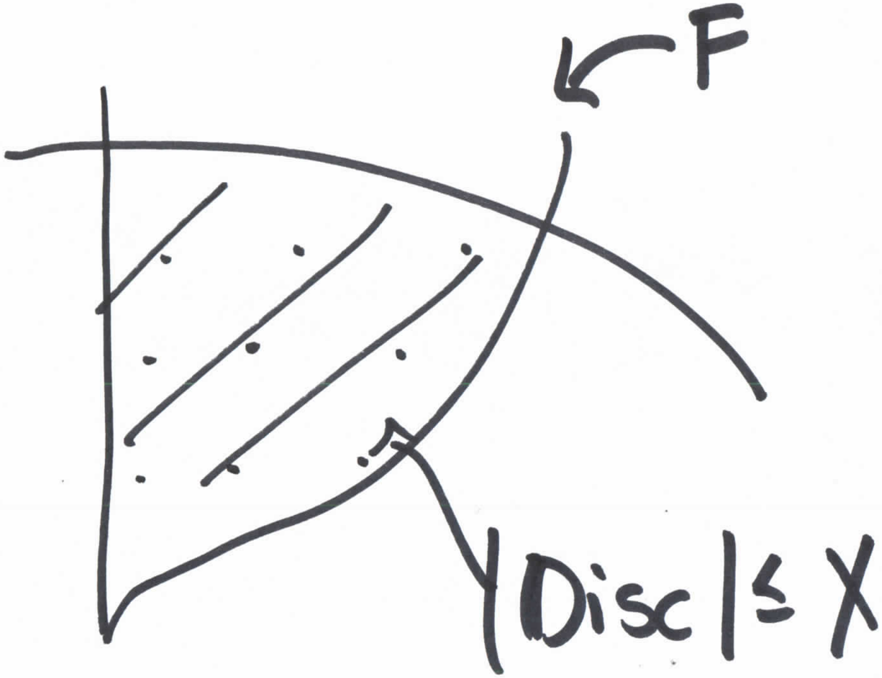
Vols: A, $1/A^2$

$N(R)$ could be

A^2 or 0

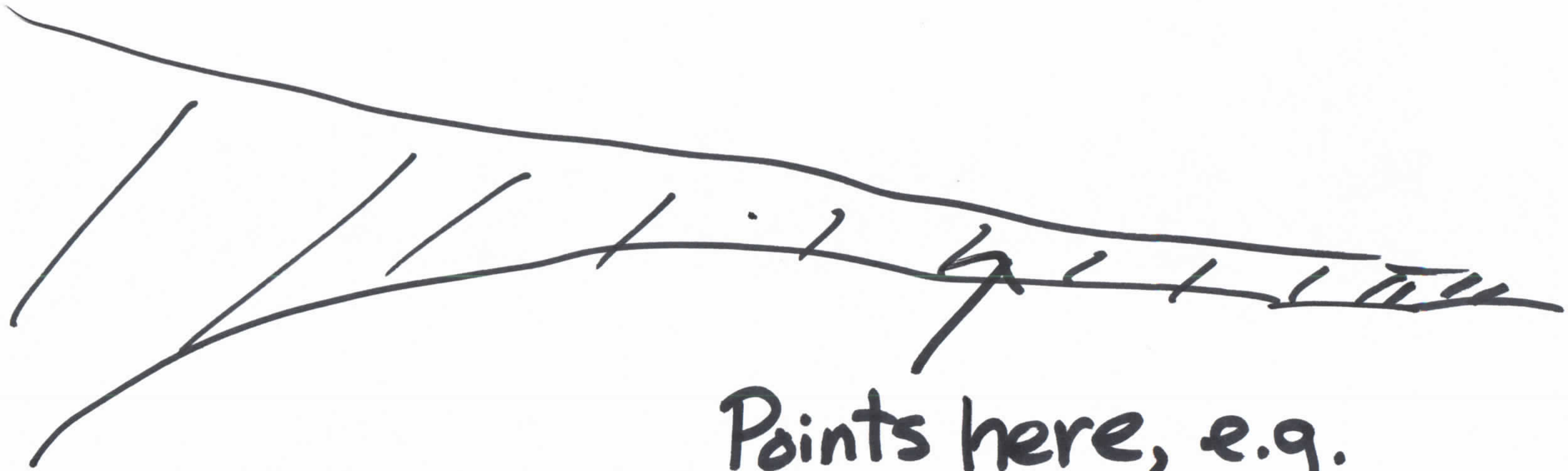
1 diml Proj

Vol: A^2, ...



$$\mathrm{Vol}(F \cap |\mathrm{Disc}| \le X)) \sim cX$$

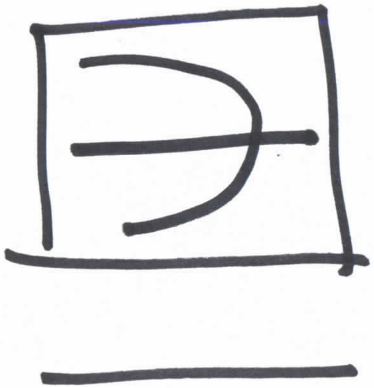
Largest Vol of a Proj $= \infty$



Points here, e.g.
with $a=0$

$\cancel{ax^3}+bx^2y+cxy^2+dy^3$

Determine we
didn't want
these points
anyway.



V_f
includes
line $x=0$

this is an "order" in $\mathbb{Q}\oplus K$ ← quadratic field

a lot of lattice pts w/ Disc = 0
e.g. corr to rings like $\mathbb{Z}[x]/_{x^3}$

Now we make a smaller region

$|\mathrm{Disc}| \geq 1$

$|a| \geq 1$

Davenport
$(|a| \geq X^{1/16})$

D-H: $N_{S_3}(X) \sim \frac{1}{3\zeta(3)} X$

Many other similar parametrizations of algebraic objects are now known.

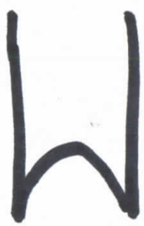
One can apply geom of #s ideas to count the objects.

Improvements to geom of #s, due to Bhargava's work on counting quartic # fields.

Finding a fundamental domain

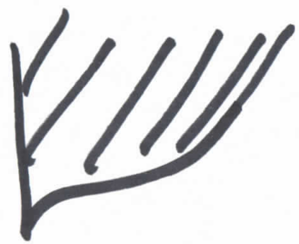
Use a fundamental domain

$\mathcal{F} = GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})$



v_0 some binary cubic form

$\mathcal{F}v_0$



fund domain for $GL_2(\mathbb{Z})$ on (a,b,c,d)

You can choose different v_0, take a lot of v_0 + get a lot fundamental domains.

Averaging over these, improves error.