Heuristic for counting number fields

$$\Gamma \subset S_n \quad \left\{ \begin{array}{l} \text{for each place } p \text{ of } \mathbb{Q}, \\ \qquad \Sigma_p \text{ set of homs } G_{\mathbb{Q}_p} \to \Gamma \end{array} \right\}^{\leftarrow \Sigma}$$

Count $G_{\mathbb{Q}} \to \Gamma$ with restriction $\underset{\overset{\uparrow}{\Sigma_p}}{G_{\mathbb{Q}_p}} \to \Gamma$

$$D_{\Gamma,\Sigma}(s) := \underset{\overset{\uparrow}{\text{constant}}}{C_\Gamma} \prod_{p \text{ place of } \mathbb{Q}} \left(\frac{1}{\#\Gamma} \sum_{\rho_p \in \Sigma_p} (\operatorname{Disc} \rho_p)^{-s} \right)$$

Check Γ abelian, Σ allows everything

Local Factor

$$\frac{1}{\#\Gamma}\sum_{\rho:\underset{\mathbb{Z}_p^*\times\langle p\rangle}{\underset{\shortparallel}{\mathbb{Q}_p^*}}\to\Gamma}(\mathrm{Disc}\,\rho)^{-s}=\sum_{\rho:\mathbb{Z}_p^*\to\Gamma}(\mathrm{Disc}\,\rho)^{-s}$$

For general Γ, heuristic/ principle is that the asymptotics of coeffs of $D_{\Gamma,\Sigma}(s)$ agree w/ asymp. of $N_{\Gamma,\Sigma}(X)$ ↖ Γ # fields satisfying Σ conds

Local factors at tame $\underset{\text{finite}}{\wedge}$ places $p \nmid \#\Gamma$

$$\frac{1}{\#\Gamma}\sum_{\rho : G_{\mathbb{Q}_p} \to \Gamma} (\operatorname{Disc} \rho)^{-s} = \frac{1}{\#\Gamma} \sum_{\substack{x, y \in \Gamma \\ xyx^{-1} = y^p}} (\operatorname{Disc} y)^{-s}$$

ρ must factor through tame <u>quotient of $G_{\mathbb{Q}_p}$</u>

generated by x, y with relation

$xyx^{-1} = y^p$. Here y generates inertia subgroup.

x lift of Frobenius

Using Disc is Artin cond of perm rep. $(\Gamma < S_n)$

$$\text{Disc } y = p^{n - \#\text{orbits}(y)}$$

Local factor $\displaystyle\frac{1}{\#\Gamma} \sum_{y \in \Gamma} \sum_{\substack{x \\ xyx^{-1} = y^p}} (\text{Disc } y)^{-s}$

$$= \frac{1}{\#\Gamma} \sum_{\substack{y \in \Gamma \\ y \sim y^p}} \frac{\#\Gamma}{\#\text{conj class } y} (\text{Disc } y)^{-s}$$

$$= \sum_{\substack{[y] \text{ conj classes} \\ \text{of } \Gamma \text{ s.t.} \\ [y]=[y]^p}} (\mathrm{Disc}\ y)^{-s}$$

always some primes

$p \equiv 1 \pmod{\#\Gamma}$

at such p

$$\sum_{\substack{[y] \text{ conj} \\ \text{class}}} (\mathrm{Disc}\ y)^{-s}$$

<u>ex</u> $\Gamma = \mathbb{Z}/2\mathbb{Z}$ $\quad 1 + p^{-s}$

$\Gamma = \mathbb{Z}/3\mathbb{Z}$ when p is 1 mod 3 all 3elts have $pz = \cancel{p}z$

p is 2 mod 3 only $z = 0$ has $pz = z$.

In particular, this heuristic predicts

1) Malle's Conj $N_\Gamma(X) \sim K_\Gamma X^{1/a(\Gamma)} (\log X)^{b(\Gamma)}$

2) Independence of local behaviors at different places.

Both have counterexamples

1) $\Gamma = C_3 \wr C_2$ $N_\Gamma(X)$ is bigger

2) abelian Γ not $(\mathbb{Z}/p\mathbb{Z})^k$ independence fails

But yet right sometimes.

$\Gamma = S_3$ Dav.-Heil. heuristic → correct ans.

$\Gamma = S_3 \subset S_6$ Bhargava-W. ✓

$\Gamma = S_4$ Bhargava ✓

$= S_5$ " ✓

$\Gamma = D_4$ Cohen-Diaz y Diaz-Oliver
order growth is right
but otherwise wrong

Davenport-Heilbronn

counting cubic # fields

parametrization of cubic rings

ring $\simeq \mathbb{Z}^3$ as an additive group

ex $[K:\mathbb{Q}]=3$ $\mathcal{O}_K$ cubic ring

$\mathbb{Z}[x]/x^3$

count cubic rings, sieve for maxl orders in fields

R cubic ring $\mathbb{Z}$-basis $1, W, T$

$$WT = q + rW + sT \qquad q, r, s \in \mathbb{Z}$$

$\omega = W - s \quad \theta = T - r$

new $\mathbb{Z}$-basis $1, \omega, \theta$ (normalized)

$$\omega\theta = n \qquad n \in \mathbb{Z}$$
$$\omega^2 = m - b\omega + a\theta \qquad m, b, a \in \mathbb{Z}$$
$$\theta^2 = \ell - d\omega + c\theta \qquad \ell, d, c \in \mathbb{Z}$$

assoc. $\Longleftrightarrow$ $n = -ad \quad m = ac \quad \ell = -bd$

Cubic rings w/ a choice of normal. basis $\longleftrightarrow$ $(a,b,c,d) \in \mathbb{Z}^4$

$\circlearrowright$ $GL_2(\mathbb{Z})$ changes basis

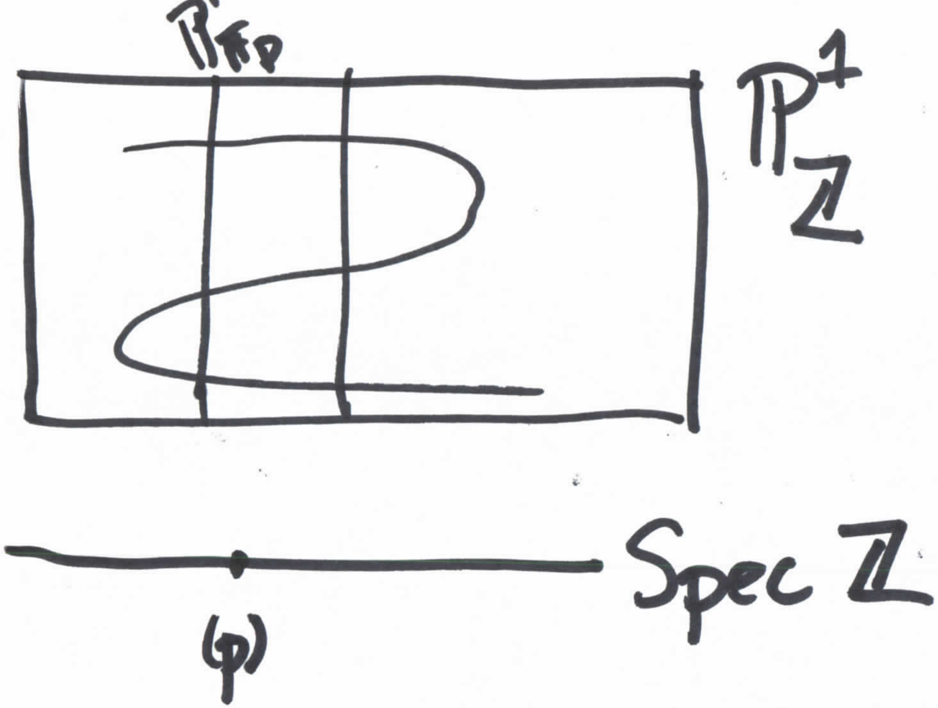
$\circlearrowright$ $GL_2(\mathbb{Z})$

cubic rings up to isom $\longleftrightarrow$ $GL_2(\mathbb{Z})$ classes of (a,b,c,d)

$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \quad g \in GL_2(\mathbb{Z})$

$$(gf)(x,y) = \frac{1}{\det(g)} f((x,y)g)$$

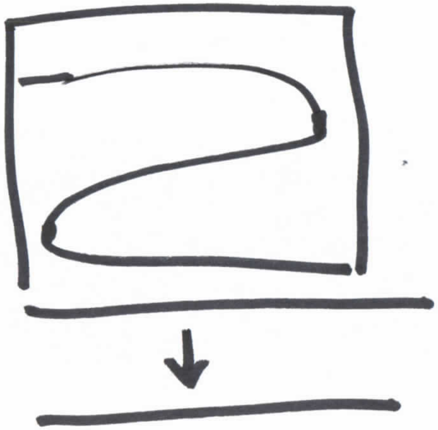
count cubic rings $\longleftrightarrow$ count $GL_2(\mathbb{Z})$ classes of binary cubic forms



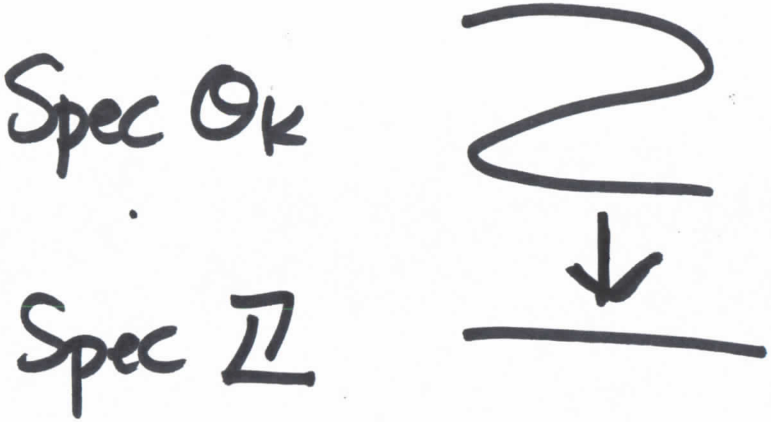
$f(x,y)$ cuts some $V_f \subset \mathbb{P}^1_{\mathbb{Z}}$

if $p \mid a,b,c,d$ get vertical fiber

if $(a,b,c,d)=1$



$\mathcal{O}_K \subset K$



When $f \not\equiv 0$, $\underbrace{\text{global functions } H^0(V_f, \mathcal{O})}_{\text{always cubic rings}}$

Thm ↙ same cubic ring as above

This cons: binary n-ic forms → rank n rings.