Counting Abelian # Fields

Count $\mathbb{Z}/3\mathbb{Z}$ fields $N_{\mathbb{Z}/3\mathbb{Z} \subset S_3}(X)$

Need to count: $J_{\mathbb{Q}} \to \mathbb{Z}/3\mathbb{Z}$

restricted, so counting

$$\prod_p \mathbb{Z}_p^* \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

For each p, what are maps

$$\mathbb{Z}_p^* \to \mathbb{Z}/3\mathbb{Z}$$

$p \neq 3$ such maps factor

$$\mathbb{Z}_p^* \to (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}/3\mathbb{Z}$$

cyclic group
order $p-1$

$p \equiv 1 \pmod 3$
3 maps

$p \equiv 2 \pmod 3$
1 map

2 non-trivial maps each have

Disc p^2

$$f(s) = \sum_{n \geq 1} a_n n^{-s} = \prod_{\substack{p \equiv 1 \\ \text{mod } 3}} (1 + 2p^{-2s})$$

$$\times (1 + 2 \cdot 3^{-4s})$$

a_n = (# cyclic cubic fieds with |disc| = n)

$\times 2$

Let χ_3 Dirchlet character mod 3

$\chi_3(1) = 1 \quad \chi_3(2) = -1$

$\zeta(2s)\,L(2s, \chi_3)$

$$= \prod_p \left(1 + p^{-2s} + \cdots\right) \prod_p \left(1 + \chi(p) p^{-2s} + \cdots\right)$$

$$= \prod_{\substack{p \equiv 1 \\ \bmod 3}} \left(1 + 2p^{-2s} + \cdots\right) \prod_{\substack{p \equiv 2 \\ \bmod 3}} \left(1 + 0p^{-2s} + \cdots\right) \prod_{p=3} (\cdots)$$

$$\frac{f(s)}{\zeta(2s)L(2s,\chi_3)}$$ abs. convergence $\operatorname{Re} s > \frac{1}{4}$

So $f(s)$ has same rightmost pole behavior as $\zeta(2s)L(2s,\chi_3)$ $\left[\begin{smallmatrix}\text{at}\\ s=\frac{1}{2}\end{smallmatrix}\right]$

Applying Tauberian thm obtain

$$N_{\mathbb{Z}/3\mathbb{Z}}(X) \sim cX^{1/2}$$ some constant c.

Local Conditions

$N_{\mathbb{Z}/2\mathbb{Z}, \Sigma}(X)$ $\qquad$ Σ split comp at 3

$J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$ $\qquad$ CFT split comp at 3

iff 1) $\mathbb{Z}_3^* \to \mathbb{Z}/2\mathbb{Z}$ trivial

2) $3 \in \mathbb{Q}_3^* \mapsto 0$

$(1, 3, 1, 1, \ldots) = (3^{-1}, 1, 3^{-1}, 3^{-1}, \ldots)$

$\uparrow J_{\mathbb{Q}}$ $\qquad$ $\uparrow$ each element is in $\mathbb{Z}_p^*$

Where does 3 go in $\mathbb{Z}_p^* \to \mathbb{Z}/2\mathbb{Z}$ when $p \neq 3$, $\neq 2$?

$$\begin{tikzcd} \mathbb{Z}_p^* \arrow[rr] \arrow[dr, two heads] & & \mathbb{Z}/2\mathbb{Z} \\ & (\mathbb{Z}/p)^* \arrow[ur] & \end{tikzcd}$$

in the non-trivial map $3 \mapsto 0$ if 3 is a $\square$ mod p

$3 \mapsto 1$ if 3 not $\square$ mod p

χ character (mod 12): $\chi(p) = \begin{cases} 1 & 3\ \square \bmod p \\ -1 & 3 \neq \square \bmod p \end{cases}$

ignore $p=2,3$

$$\frac{1}{2}\left(\prod_p (1+p^{-s}) + \prod_p \left(1+\chi(p)p^{-s}\right)\right)$$

counted $J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$

counts $J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$ w/ sign + if $(3,1,3,3,\ldots) \mapsto 0$

sign − $(3,1,3,3,\ldots) \mapsto 1$

counts quad fields s.c. @ 3

so apply Tauberian theorem

first Euler product gives rightmost pole $s=1$

can ignore seccond Euler product b/c it is analytic ~~past~~ on $\operatorname{Re} s \geq 1$

In general, any abelian G, any local conditions Σ (finitely many places)

Write

$$f(s) = \sum \text{Euler products}$$

Dirchlet series

identify rightmost poles all Euler products

$N_{G,\Sigma}$

For example, independence at different primes <u>fails</u> unless $G = (\mathbb{Z}/p\mathbb{Z})^k$

Grunwald-Wang

There are local extensions that never happen from a global extension.

There is no $\mathbb{Z}/8\mathbb{Z}$ extension K of $\mathbb{Q}$ for which K_2 is an unramified extn of $\mathbb{Q}_2$ of deg 8 (equiv. in which 2 is totally inert).

Over K number field, also can have local extns at diff places that can each occur from global extn, but can't occur together.

Bad reasons can be eliminated if instead of counting by discriminant, we count by conductor. ← class field theory

product of ram primes to some powers

When counting by conductor, you have nice behavior, independence except for <u>when</u> G-W interfere.

So what should we expect when counting general number fields?

(Open question: what should we be counting them by?)

Heuristic

Count $G_{\mathbb{Q}} \to G \subset S_n$

CFT G abelian $G_{\mathbb{Q}} \to G$ built out of maps $G_{\mathbb{Q}_p} \to G$ in some way

asym. behaves as if $G_{\mathbb{Q}} \to G$ (nonab G) built out of $G_{\mathbb{Q}_p} \to G$ in a similar way