Relation between counting # fields & class group averages

Let K be imag. quadratic field

$\mathrm{Sur}(\mathrm{C}\ell(K), \mathbb{Z}/3\mathbb{Z})$ by class field theory

correspond to

$$\begin{tikzcd} L \arrow[d, dash, "\text{unram, } \mathbb{Z}/3\mathbb{Z}\text{-extns}"] \\ K \arrow[d, dash, "2"] \\ \mathbb{Q} \end{tikzcd}$$

Also, CFT tells us $L/\mathbb{Q}$ Galois, with $\mathrm{Gal}(L/\mathbb{Q}) = S_3 < S_6$ (regular)

Conversely, any $L/\mathbb{Q}$ S_3 Galois sextic w/ quad subfield K with L/K unram arises this way.

Since L/K unram, $|\mathrm{Disc}\, L| = |\mathrm{Disc}\, K|^3$

Also

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, dash] \arrow[dr, dash] & \\ K_3 \arrow[dr, dash, "3"'] & & K \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

to translate this into a question about counting non-Galois cubics

Galois Permutation Representations

For a field F, let $G_F := \mathrm{Gal}(\overline{F}/F)$

$$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

étale F-algebras are direct sums of finite (separable) field extensions of F

$$\left\{ G_F \to S_n \right\}_{\text{isom.}} \xleftrightarrow{\text{bij.}} \left\{ \begin{array}{l} \text{degree } n \\ \text{étale } F\text{-algebras} \end{array} \right\}$$

transitive (single orbit) $\longleftarrow - - - \longrightarrow$ field extensions

$K/\mathbb{Q}$ degree n # field

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \to S_n$$

action of $G_{\mathbb{Q}}$
of n maps $K \to \overline{\mathbb{Q}}$

M degree n étale $\mathbb{Q}$-algebra

n maps $M \to \overline{\mathbb{Q}}$

ex $M = K_1 \oplus K_2 \oplus K_3$ $\quad K_i$ deg n_i $\quad \sum n_i = n$

n maps $M \to \overline{\mathbb{Q}}$ are n_1 maps $K_1 \to \overline{\mathbb{Q}}$ $\quad K_2, K_3 \to 0$

n_2 maps $K_2 \to \overline{\mathbb{Q}}$ $\quad K_1, K_3 \to 0$

n_3 maps $K_3 \to \overline{\mathbb{Q}}$ $\quad K_1, K_2 \to 0$

$G_{\mathbb{Q}} \circlearrowright$

in ex. $G_{\mathbb{Q}} \to S_n$ has 3 orbits

Given $G_{\mathbb{Q}} \to S_n$

let $H_i \subset G_{\mathbb{Q}}$ be the stabilizer of i, as i runs over one ~~an~~ element from each orbit

$\updownarrow$ Galois theory

correspond to

$K_i/\mathbb{Q}$

étale extn $\bigoplus K_i$

- You can recover the discriminant of an étale extn from $G_{\mathbb{Q}} \to S_n$

Tauberian Theorem

<u>**Thm**</u> Let $f(s) = \sum_{n \geq 1} a_n \cdot n^{-s}$ with $f(s)$ absolute convergent for $\mathrm{Re}(s) > 1$, and with meromorphic ~~analytic~~ continuation to $\mathrm{Re}(s) \geq 1$.
If f has a simple pole at $s=1$ with residue r, then

$$\sum_{1 \leq n \leq X} a_n = rX + o(X).$$

In <u>general</u>

- finding a meromorphic continuation of a Dirchlet series that you wrote to counts some things can range from hard to impossible.
- even with a continuation, analysis of the poles can be hard

(e.g. $a_n = \#$ quartic (or quintic) $\#$ fields of $|\text{disc}| = n$)

Counting Abelian Number Fields

using class field theory

$$J_{\mathbb{Q}} = \Big(\prod_{\substack{p \\ \text{places}}}{}' \mathbb{Q}_p^*\Big) \Big/ \mathbb{Q}^*$$

CFT:

$G_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z} = S_2$ correspond to $J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$

Count: $\phi : J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$

ϕ restricts to $\phi_0 : \prod_p \mathbb{Z}_p^* \to \mathbb{Z}/2\mathbb{Z}$

($\mathbb{Z}_\infty^*$ to denote positive reals)

Any $\phi_0 : \prod_p \mathbb{Z}_p^* \to \mathbb{Z}/2\mathbb{Z}$ <u>extends</u> <u>uniquely</u> to a $J_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$

Need to know where

$(1, 1, .. 1, p, 1, .. 1)$ goes

in $\mathbb{Q}_p^*$ place

$(p^{-1}, p^{-1}, .. p^{-1}, 1, p^{-1}, .. p^{-1})$ goes

Now counting $\prod_p \mathbb{Z}_p^* \to \mathbb{Z}/2\mathbb{Z}$

i.e. $\mathbb{Z}_p^* \to \mathbb{Z}/2\mathbb{Z}$ for each p

$p \neq 2, \infty$ 2 such maps 1 trivial
1 not $\leftarrow$ Disc p

$$\mathbb{Z}_p^* \to (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}/2\mathbb{Z}$$

$p=2$ 4 maps

Disc 1, 4, 8, 8

$p=\infty$ 1 map

Dirchlet series

$$\left(\prod_{\substack{p\neq 2 \\ \text{primes}}} (1+p^{-s})\right)(1+4^{-s}+8^{-s}+8^{-s})$$

$a_n = \#$ quad fields
$|\mathrm{Disc}| = n$

$$\| $$

$$\frac{(1+4^{-s}+2\cdot 8^{-s})}{(1+2^{-s})} \quad \frac{\zeta(s)}{\zeta(2s)} \Rightarrow N_{S_2}(X) \sim \frac{6}{\pi^2} X.$$

mero ctn ✓
poles rightmost ✓