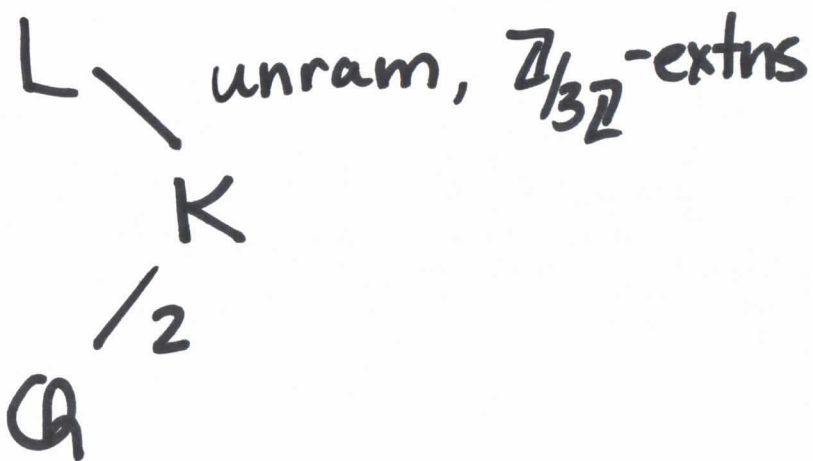


Relation between counting # fields  
+ class group averages

Let  $K$  be imag. quadratic field

$\text{Sur}(\text{Cl}(K), \mathbb{Z}/3\mathbb{Z})$  by class field theory

correspond to

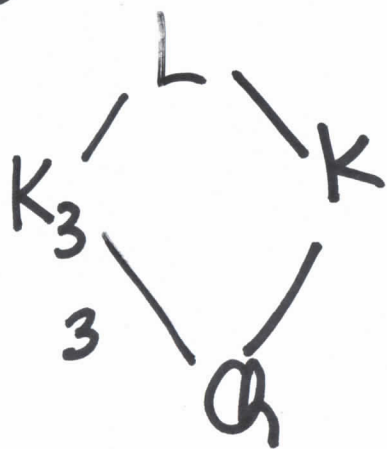


Also, CFT tells  
us  $L/\mathbb{Q}$  Galois,  
with  $\text{Gal}(L/\mathbb{Q}) = S_3 \subset S_6$   
(regular)

Conversely, any  $L/\mathbb{Q}$   $S_3$  Galois sextic  
w/ quad subfield  $K$  with  $L/K$  unram  
arises this way.

Since  $L/K$  unram,  $|\text{Disc } L| = |\text{Disc } K|^3$

—  
Also



to translate this  
into a question  
about counting  
non-Galois cubics

# Galois Permutation Representations

For a field  $F$ , let  $G_F := \text{Gal}(\overline{F}/F)$

$$G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

étale  $F$ -algebras are direct sums  
of finite (separable)  
field extensions of  $F$

$$\left\{ G_F \rightarrow S_n \right\}_{\text{isom.}} \xleftrightarrow{\text{bij.}} \left\{ \begin{array}{l} \text{degree } n \\ \text{étale } F\text{-algebras} \end{array} \right\}$$

transitive (single orbit)  $\longleftrightarrow$  field extensions

$K/\mathbb{Q}$  degree  $n$  # field

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Gal}(\widehat{K}/\mathbb{Q}) \rightarrow S_n$$

action of  $G_{\mathbb{Q}}$   
of  $n$  maps  $K \rightarrow \overline{\mathbb{Q}}$

---

$M$  degree  $n$  étale  $\mathbb{Q}$ -algebra

$n$  maps  $M \rightarrow \overline{\mathbb{Q}}$

ex  $M = K_1 \oplus K_2 \oplus K_3$   $K_i$  deg  $n_i$   $\sum n_i = n$

$n$  maps  $M \rightarrow \overline{\mathbb{Q}}$  are  $n_1$  maps  $K_1 \rightarrow \overline{\mathbb{Q}}$   $K_2, K_3 \rightarrow 0$

$n_2$  maps  $K_2 \rightarrow \overline{\mathbb{Q}}$   $K_1, K_3 \rightarrow 0$

$n_3$  maps  $K_3 \rightarrow \overline{\mathbb{Q}}$   $K_1, K_2 \rightarrow 0$


$G_{\mathbb{Q}}$

in ex.  $G_{\mathbb{Q}} \rightarrow S_n$  has 3 orbits

Given  $G_{\mathcal{O}} \rightarrow S_n$

let  $H_i \subset G_{\mathcal{O}}$  be the stabilizer  
of  $i$ , as  $i$  runs  
over ~~the~~ one element  
from each orbit

Galois  
theory



correspond to

$K_i/\mathcal{O}$

étale extn  $\bigoplus K_i$

- You can recover the discriminant of an étale extn from  $G_{\mathcal{O}} \rightarrow S_n$



# Tauberian Theorem

Thm Let  $f(s) = \sum_{n \geq 1} a_n \cdot n^{-s}$  with

$f(s)$  <sup>absolute</sup> convergent for  $\operatorname{Re}(s) > 1$ , and with  
<sup>meromorphic</sup> ~~analytic~~ continuation to  $\operatorname{Re}(s) \geq 1$ .

If  $f$  has a simple pole at  $s=1$  with residue  $r$ , then

$$\sum_{1 \leq n \leq x} a_n = r x + o(x).$$

In general

- finding a meromorphic continuation of a Dirichlet series that you wrote to count some things can range from hard to impossible.
- even with a continuation, analysis of the poles can be hard  
(e.g.  $a_n = \# \text{quartic (or quintic) fields of } |\text{disc}|=n$ )

# Counting Abelian Number Fields using class field theory

$$J_{\mathbb{Q}} = \left( \prod_{\substack{P \\ \text{places}}} \mathbb{Q}_P^* \right) / \mathbb{Q}^*$$

CFT:

$$G_{\mathbb{Q}} \rightarrow \mathbb{Z}/2\mathbb{Z} = S_2$$

correspond to

$$J_{\mathbb{Q}} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

$$\text{Count: } \phi: J_{\mathbb{Q}} \rightarrow \mathbb{Z}/2\mathbb{Z}$$



$\phi$  restricts to  $\phi_0: \prod_p \mathbb{Z}_p^* \rightarrow \mathbb{Z}/2\mathbb{Z}$

( $\mathbb{Z}_\infty^*$  to denote positive reals)

Any  $\phi_0: \prod_p \mathbb{Z}_p^* \rightarrow \mathbb{Z}/2\mathbb{Z}$  extends uniquely to  
a  $J\mathbb{Q} \rightarrow \mathbb{Z}/2\mathbb{Z}$

Need to know where

$(1, 1, \dots, 1, p, 1, \dots, 1)$  goes

$\uparrow$   
in  $\mathbb{Q}_p^*$  place

$(p^{-1}, p^{-1}, \dots, p^{-1}, 1, p^{-1}, \dots, p^{-1})$  goes



# Dirichlet series

$$\left( \prod_{\substack{p \neq 2 \\ \text{primes}}} (1 + p^{-s}) \right) (1 + 4^{-s} + 8^{-s} + 8^{-5})$$

$a_n = \#$  quad fields  
 $|\text{Disc}| = n$

$$\frac{(1 + 4^{-s} + 2 \cdot 8^{-s})}{(1 + 2^{-s})} \frac{\zeta(s)}{\zeta(2s)} \Rightarrow N_{S_2}(X) \sim \frac{6}{\pi^2} X.$$

mero ctn ✓  
poles rightmost ✓