2014 ARIZONA WINTER SCHOOL COURSE AND PROJECT OUTLINE

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1. Course outline

This course will consist of two roughly independent topics.

1.1. **Bertini theorems and the closed point sieve.** The classical Bertini theorems over an infinite field k state that if a subscheme $X \subseteq \mathbb{P}^n_k$ has a certain property (smooth, geometrically irreducible, geometrically reduced), then a sufficiently general hyperplane section over k has the property too. If k is finite, however, such statements can fail: for example, if X is smooth, it can happen that all of the finitely many hyperplanes H in $\mathbb{P}^n_k$ are tangent to X; in this case $H \cap X$ is never smooth.

The paper [Poo04] proved a Bertini smoothness theorem over finite fields, in which hyperplanes were replaced by hypersurfaces of degree d tending to ∞. For fixed d, consider the probability p_d that the intersection of a *random* degree d hypersurface H with the given smooth X is smooth; the result is that $\lim_{d\to\infty} p_d$ is a special value of the zeta function of X, and in particular is positive.

Here is the idea. Smoothness can be tested one closed point at a time.

At a degree e closed point x of X, if d is large enough, then the probability that $H \cap X$ is singular at x turns out to be $q^{-e(m+1)}$, where $m := \dim X$. Heuristically, these conditions at different x are independent, so after sieving out such H for all closed points $x \in X$, the fraction remaining should be $\prod_{\text{closed } x\in X} \left(1 - q^{-e(m+1)}\right)$. The hard part is to make this rigorous even though infinitely many x are involved.

More recently, the paper [CP13] proved a Bertini irreducibility theorem over finite fields. This can no longer be done with a sieve over closed points, since irreducibility cannot be tested locally, but ultimately it again boils down to a counting problem: how many nontrivial sums of effective divisors $D_1 + D_2$ are there on X resulting in a hypersurface section?

The course will explain how to use these techniques, with an eye towards the open problems in the project.

1.2. **Selmer group heuristics.** Given an elliptic curve E over a global field k, and a positive integer n, the n-Selmer group $\operatorname{Sel}_n E$ is a computable finite abelian group that provides an

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upper bound for $E(k)/nE(k)$. More precisely, there is an exact sequence

$$0 \to \frac{E(k)}{nE(k)} \to \operatorname{Sel}_n E \to Ш[n] \to 0,$$

where Ш is the Shafarevich–Tate group of E and $Ш[n] := \{x \in Ш : nx = 0\}$. Letting n run through powers of a prime p and taking the direct limit leads to an exact sequence

$$0 \to E(k) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \to \operatorname{Sel}_{p^\infty} E \to Ш[p^\infty] \to 0. \tag{1}$$

Whereas the order of $\operatorname{Sel}_n E$ was only an upper bound for $n^{\operatorname{rk} E(k)}$, the structure of the group $\operatorname{Sel}_{p^\infty} E$ determines the rank of $E(k)$ exactly, if one assumes the conjecture that Ш is finite.

What is the distribution of $\operatorname{Sel}_n E$ among abelian groups as E varies over, say, all elliptic curves over k? What is the distribution of (1) among all short exact sequences of abelian groups? There is now a conjectural answer to both these questions [PR12, BKL+13], compatible with the few theorems that have been proved in special cases [HB93, HB94, dJ02, SD08, Kan13, BS10a, BS10b], and compatible with other conjectures that have been made over the years [Gol79, KS99a, KS99b, Del01, Del07, DJ13]. The heuristic also bears a connection to the Cohen–Lenstra heuristics [CL84] as reinterpreted in [FW89] and [VE10, Section 4.1].

2. Project

The project will be to develop new Bertini-type results over finite fields:

(1) Generalize the Bertini smoothness theorem to a setting where the given smooth variety X is defined over a finite field that is larger than the field over which the hypersurfaces are taken. (The analogue for the Bertini irreducibility theorem is done in [CP13], and some partial results for smoothness are there too.)
(2) The Bertini irreducibility theorem over finite fields states the fraction of bad hypersurfaces tends to 0 as the degree d tends to ∞ [CP13]. Can one refine the counting argument to obtain an explicit upper bound in terms of d, or perhaps even an asymptotic formula?
(3) Prove a Bertini irreducibility theorem over finite fields in the setting where the hypersurface is required to contain a certain subvariety. That is, combine [CP13] and [Poo08].
(4) Prove a "semiample version" of the Bertini irreducibility theorem over finite fields. That is, combine [CP13] and [EW12].
(5) Formulate and prove a Bertini reducedness theorem over finite fields. (For the Bertini reducedness theorem over *infinite* fields, see [Jou83, Théorème 6.3(3)].)

Prerequisite: At least a semester of graduate-level algebraic geometry, including familiarity with the language of schemes.

References

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