Last time:

$$\begin{tikzcd}
 & & \mathrm{Sel}_n E \arrow[d, phantom, "\subset" rotate=-90] & \\
0 \arrow[r] & \dfrac{E(\mathbb{Q})}{nE(\mathbb{Q})} \arrow[r] \arrow[d] & H^1(\mathbb{Q}, E[n]) \arrow[r] \arrow[d, "\beta"] & H^1(\mathbb{Q}, E) \arrow[d, "\gamma"] \\
0 \arrow[r] & \dfrac{E(\mathbb{A})}{nE(\mathbb{A})} \arrow[r, "\alpha"] & H^1(\mathbb{A}, E[n]) \arrow[r] & H^1(\mathbb{A}, E)
\end{tikzcd}$$

$\mathrm{Sel}_n E := \beta^{-1}(\mathrm{im}\,\alpha)$ ← intersection of max. isotropic subspaces (for $n=p$)

$Ш := \ker \gamma$ ← torsion ab. group, conjecturally finite

$$0 \to \frac{E(\mathbb{Q})}{nE(\mathbb{Q})} \to \mathrm{Sel}_n E \to Ш[n] \to 0$$

$n = p^e$

$\varinjlim_e$

$$0 \to E(\mathbb{Q}) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \to \mathrm{Sel}_{p^\infty} E \to Ш[p^\infty] \to 0$$

$\leftarrow \mathrm{Seq}_E$

Each term is a $\mathbb{Z}_p$-module of the form

$$\left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right)^s \oplus \text{finite}.$$

The OGr returns!

Last time: $\mathrm{OGr}_n(\mathbb{F}_p) := \left\{ \begin{array}{c} \text{max. isot. subspaces of} \\ V := \mathbb{F}_p^{2n} \\ Q := x_1y_1 + \ldots + x_ny_n \end{array} \right\}$

$\cap$

$\mathrm{Gr}_{n,2n}(\mathbb{F}_p) := \left\{ \begin{array}{c} \text{all } n\text{-dim subspaces} \\ \text{of } \mathbb{F}_p^{2n} \end{array} \right\}$

More generally,

$$\mathrm{Gr}_{n,m}(A) = \left\{ \begin{array}{c} \text{loc. free rank } n \text{ } A\text{-submodules} \\ Z \leq A^m \\ \text{s.t. } Z \text{ is a direct summand} \end{array} \right\}$$

A: any comm. ring

$$\mathrm{OGr}_n(A) := \left\{ Z \in \mathrm{Gr}_{n,2n}(A) : Q|_Z = 0 \right\}$$

$Gr_{m,m}$ and OGr_n are repr.
by smooth projective schemes$/\mathbb{Z}$

OGr_n has 2 connected components

Fact: For any field k,
$Z, Z' \in OGr_n(k)$ are in the same component
$\iff \dim(Z \cap Z') \equiv n \pmod 2$

smoothness $\Rightarrow$ the fibers of $OGr_n\left(\frac{\mathbb{Z}}{p^{e+1}\mathbb{Z}}\right) \to OGr_n\left(\frac{\mathbb{Z}}{p^e\mathbb{Z}}\right)$
have constant size

$\therefore$ Get "uniform" prob. measure on

$$OGr_n(\mathbb{Z}_p) = \varprojlim_e OGr_n(\mathbb{Z}/p^e\mathbb{Z})$$

Model

$V := \mathbb{Z}_p^{2n}$ Fix $W := \mathbb{Z}_p^n \times 0$

Choose random $Z \in \mathrm{OGr}_n(\mathbb{Z}_p)$. in $V \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}$

Form

$$0 \to \underbrace{(Z \cap W) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}}_{R} \to \underbrace{\left(Z \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right) \cap \left(W \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right)}_{S} \to T \to 0$$

$$T = S/R$$

<u>Thm.</u> (Bhargava, Kane, Lenstra, Poonen, Rains)

$$\lim_{n \to \infty} \left(\text{distr. of } 0 \to R \to S \to T \to 0 \right) \text{ exists.}$$

<u>Conj.</u> (BKLPR): The limit distr. equals the distr. of Seq_E for $E \in \mathcal{E}$.

Consequences for rank

$$(Z \cap W) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \simeq \left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right)^r$$

where $r := \dim_{\mathbb{Q}_p} \left(Z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) \cap \left(W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)$

$$\approx \begin{cases} 0, & \text{for } Z \text{ is one comp. of OGr} \\ 1, & \text{------------ other ------} \end{cases}$$

outside a lower-dim locus in OGr (measure 0)

~~Conj.~~ Cor. Conj. $\Rightarrow$ $\begin{cases} 50\% \text{ of ell. curves have rank } 0 \\ 50\% \text{ ———————— } 1 \\ 0\% \text{ ———————— } \geq 2 \end{cases}$

Consequences for Sel_{p^e}

Fix p.

If $\underbrace{E[p](\mathbb{Q}) = 0}_{\substack{\text{true for 100\% of } E \\ \text{by Hilbert irreducibility theorem}}}$, then $\mathrm{Sel}_{p^e} E = (\mathrm{Sel}_{p^\infty} E)[p^e]$

__Cor.__ Conj $\Rightarrow$ $$(\text{distr. of } \mathrm{Sel}_{p^e} E) = \lim_{n\to\infty} \left(\begin{array}{c} \text{distr. of } Z \cap W \\ Z, W \in \mathrm{OGr}_n\left(\mathbb{Z}/p^e\mathbb{Z}\right) \end{array} \right)$$

__~~Conj.~~ Thm__ Exp. number of inj. homoms $\left(\mathbb{Z}/p^e\mathbb{Z}\right)^m \longrightarrow Z \cap W$

$$= (p^e)^{m(m+1)/2}.$$

Consequences for $Ш$

R = max. divisible subgp. of S

T is finite

Cor. Conj. $\Longrightarrow$ $Ш[p^\infty]$ is finite for 100% of E

Condition on rank $E(\mathbb{Q})$.

Three distr. on $\{$finite abelian p-gps.$\}$,

each conjectured to be the distr. of $Ш[p^\infty]$

for $E \in \mathcal{E}$ of rank r.

① Delaunay : The distr. in which

$$\mathrm{Prob}(G) := \frac{\# G^{1-r}}{\# \mathrm{Aut}(G, [\ ,\ ])} \prod_{i=r+1}^{\infty} \left(1 - p^{1-2i}\right)$$

any nondeg alt. pairing

$[\ ,\ ] : G \times G \to \mathbb{Q}_p / \mathbb{Z}_p$

(if $[\ ,\ ]$ does not exist,

$\mathrm{Prob}(G) = 0$)

② BKLPR :

Choose random $A \in M_{2n+r}(\mathbb{Z}_p)$ such that

$$A^T = -A$$

$$\operatorname{rank} A = 2n$$

$$\mathbb{Z}_p^{2n+r} \xrightarrow{A} \mathbb{Z}_p^{2n+r} \to (\operatorname{coker} A) \to 0$$

Take ~~the~~ $\lim_{n\to\infty}$ (distr. of $(\operatorname{coker} A)_{\mathrm{tors}}$).

③ BKLPR : Choose random $Z \in OGr_n(\mathbb{Z}_p)$ s.t. $\operatorname{rank}(Z \cap W) = r$.

Form $0 \to R \to S \to T \to 0$.

Take $\lim_{n\to\infty}$ (distr. of T).

<u>Thm.</u> (BKLPR) For each $r \geq 0$, these distr. coincides.