Bertini smoothness theorem over $\mathbb{F}_q$

$X \subset \mathbb{P}^n$ over $\mathbb{F}_q$

smooth of dim m

quasi-proj. subscheme

$$\mathcal{P} := \{ f \in S_{\text{homog}} : H_f \cap X \text{ is smooth of dim } m-1 \}$$

Then

$$\mu(\mathcal{P}) = \zeta_X(m+1)^{-1}.$$

Proof of theorem for $X = \mathbb{A}^2$ in $\mathbb{P}^2$:

Identify $f \in S_{homog}$ with

$$\bar{f} = f(1,x,y) \in \mathbb{F}_q[x,y]$$

$f \in \mathcal{P} \iff H_f$ is smooth [of dim 1] at each closed pt. $P \in \mathbb{A}^2$

For each P,

H_f is smooth at $P \iff f(P), \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)$ are not all zero

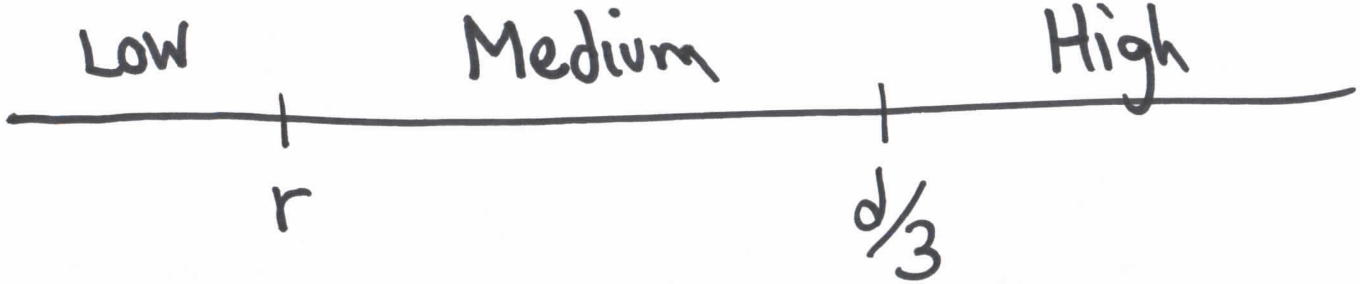
in $\kappa(P) = \mathbb{F}_{q^{\deg P}}$

Fake proof:

$$\mathrm{Prob}(H_f \text{ is smooth at } P) = 1 - \frac{1}{q^{3\deg P}}$$

$$\therefore \mathrm{Prob}\left(H_f \text{ is smooth at all } P \in \mathbb{A}^2\right) = \prod_{P \in \mathbb{A}} \left(1 - \frac{1}{q^{3\deg P}}\right)$$

$$= \zeta_{\mathbb{A}^2}(3)^{-1}.$$



① Low degree

$$\mathcal{P}_r := \left\{ f \in S_{\text{homog}} : H_f \text{ is smooth at } P \text{ for all } P \in \mathbb{A}^2 \text{ of } \deg \leq r \right\}$$

<u>Lemma</u> $$\mu(\mathcal{P}_r) = \prod_{\substack{P \in \mathbb{A}^2 \\ \deg P \leq r}} \left(1 - \frac{1}{q^{3 \deg P}} \right).$$

<u>Proof</u>: $\mathfrak{m}_P \subseteq \mathbb{F}_q[x,y]$

max. ideal corresponding to P

$$I := \prod_{\deg P \le r} \mathfrak{m}_P^2$$

Then $f \in S_d$ belongs to $\mathcal{P}_r \iff$ the image of f under

$$\mathbb{F}_q[x,y]_{\le d} \xrightarrow{\phi_d} \frac{\mathbb{F}_q[x,y]}{I} \simeq \prod_{\deg P \le r} \frac{\mathbb{F}_q[x,y]}{\mathfrak{m}_P^2}$$

is nonzero in each factor

ϕ_d is surjective if d is large.

How large?

Let $V_d := \operatorname{im} \phi_d$.

Then $V_{d+1} = V_d + xV_d + yV_d$

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_D = V_{D+1} = \cdots$$

1-dim

$$D \leq \dim_{\mathbb{F}_q} \frac{\mathbb{F}_q[x,y]}{I}$$

Thus ϕ_d is surjective for $d \geq \dim_{\mathbb{F}_q} \frac{\mathbb{F}_q[x,y]}{I}$.

② Medium degree

$$Q_r := \bigcup_d \left\{ f \in S_d : \exists P \text{ with } r < \deg P \le \frac{d}{3} \text{ at which } H_f \text{ is not smooth} \right\}$$

Lemma: $\bar{\mu}(Q_r) \to 0$ as $r \to \infty$.

Proof: $\mathbb{F}_q[x,y]_{\le d} \twoheadrightarrow \dfrac{\mathbb{F}_p[x,y]}{\mathfrak{m}_p^2}$ since $d \ge 3 \deg P$

so $\bar{\mu}(Q_r) \le \limsup_{d \to \infty} \sum_{r < \deg P \le \frac{d}{3}} \dfrac{1}{q^{3 \deg P}} \to 0$ as $r \to \infty$.

Lemma $Z \subseteq \mathbb{A}^2$ $\dim Z \geq 1$

$$\frac{\#\{f \in \mathbb{F}_q[x,y]_{\leq d} : f|_Z = 0\}}{\# \mathbb{F}_q[x,y]_{\leq d}} \leq q^{-d}$$

Proof: Choose a cordinate, say x, such that x is nonconstant on Z.

$$0 \to \mathrm{Num.} \longrightarrow \mathbb{F}_q[x,y]_{\leq d} \longrightarrow \text{functions on } Z$$

$1, x, x^2, x^3, \dots, x^d$

③ High degree

$$\mathcal{R} = \bigcup_d \left\{ f \in S_d : \exists P \text{ with } \deg P > \frac{d}{3} \text{ at which } H_f \text{ is not smooth} \right\}$$

Lemma: $\mu(\mathcal{R}) = 0$.

Proof:

$$f := f_0 + g_1^p x + g_2^p y + h^p$$

for random f_0, g_1, g_2, h of degrees $\leq d, \leq \frac{d-1}{p}, \leq \frac{d-1}{p}, \leq \frac{d}{p}$ is a random elt. of $\mathbb{F}_q[x,y]_{\leq d}$.

Then

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f_0}{\partial x} + g_1^p \\ \frac{\partial f}{\partial y} &= \frac{\partial f_0}{\partial y} + g_2^p \end{aligned}}$$

Given f_0, there is at most one g_1 such that $\frac{\partial f}{\partial x} \doteq 0$.

(i) Prob $\Big($ conditioned on a choice of f_0 that g_1 is such that $\dim\left\{\frac{\partial f}{\partial x} = 0\right\} \leq 1 \Big) \doteq 1$

(ii) Prob $\Big($ ——— f_0, g_1 ——— g_2 ——— $\dim\left\{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\right\} \leq 0 \Big) \doteq 1$

fails $\iff$ $\frac{\partial f_0}{\partial y} + g_2^p$ vanishes on some comp. of $\left\{\frac{\partial f}{\partial x} = 0\right\}$

Prob $\approx d q^{-d}$

(iii) Prob $\Big($ ——— f_0, g_1, g_2 that h is such that $\left\{f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\right\}$ has no pts. of deg $> \frac{d}{3} \Big) \doteq 1$

End of proof:

$$\mathcal{P} = \mathcal{P}_r \setminus \mathcal{Q}_r \setminus R$$

As $r \to \infty$

$$\mu(\mathcal{P}_r) \to \zeta_{\mathbb{A}^2}(3)^{-1}$$

$$\bar{\mu}(\mathcal{Q}_r) \to 0$$

$$\mu(R) = 0.$$

So $\mu(\mathcal{P}) = \zeta_{\mathbb{A}^2}(3)^{-1}$.

Applications / Variants

① same thm., but prescrible Taylor coefficients at finitely many P

② Space-filling curves: Given nice $X/_{\mathbb{F}_q}$ of $\dim \geq 1$,

smooth, proj., geom. int.

$\exists$ nice curve $Y \subseteq X$ passing through all pts. in $X(\mathbb{F}_q)$.

③ Ab. vars. as quotients of Jacobians

Given $A/_{\mathbb{F}_q}$ of $\dim \geq 1$,

$\exists$ nice curve $X \subset A$ s.t. $\mathrm{Jac}\, X \to A$ is surjective.

Proof: Find X passing through all ℓ-torsion pts. of A.

Then $\operatorname{im}(\mathrm{Jac}\, X \to A)$ has $\ell^{2\dim A}$ ℓ-torsion pts.

Nghi Nguyen's thesis:

<u>Whitney embedding theorem / $\mathbb{F}_q$.</u>

X nice curve $/\mathbb{F}_q$

Then $\exists$ closed immersion $X \hookrightarrow \mathbb{P}^3_{\mathbb{F}_q}$

$\Longleftrightarrow$ for each ~~curve~~ $e \geq 1$

closed pts. of deg e on X

$\leq$ # " ———— " $\mathbb{P}^3$