PART 1: Sieves in arithmetic geometry

Warmup 1: squarefree integers

Fix $\mathcal{P} \subseteq \mathbb{N}$

$$\underset{\text{density}}{\operatorname{Prob}(\mathcal{P})} := \lim_{B \to \infty} \frac{\#\mathcal{P} \cap [1, B]}{B} \qquad \overline{\operatorname{Prob}} := \limsup$$

Thm. $\operatorname{Prob}(n \text{ is squarefree}) = \prod_p (1 - p^{-2}) = \zeta(2)^{-1} = \dfrac{6}{\pi^2}.$

Proof:

① small primes

$$\mathrm{Prob}(p^2 \nmid n \text{ for all } p \le r) = \prod_{p \le r} \left(1 - \frac{1}{p^2}\right)$$

$$\overline{\mathrm{Prob}}(n \text{ is divisible by } p^2 \text{ for some } p > r) \to 0$$

as $r \to \infty$

$$\limsup_{B \to \infty} \frac{\#\{n \le B : n \text{ is divisible by } p^2 \text{ for some } p > r\}}{B} \to 0$$

as $r \to \infty$

② Medium-sized primes

$$\#\{n \le B : n \text{ is div. by } p^2 \text{ for some } p \in (r, \sqrt{B}]\}$$

$$\le \sum_{p \in (r, \sqrt{B}]} \lfloor B/p^2 \rfloor \le \sum_{p \in (r, \sqrt{B})} \frac{B}{p^2} \le B \int_r^\infty \frac{1}{x^2}\,dx$$

$$= B/r.$$

③ Large primes

$\#\{n \le B : n \text{ is div. by } p^2 \text{ for some } p > \sqrt{B}\}$

$= 0.$

<u>Warmup 2</u> : sqfree values of a polynomial

Prob(n^4+1 is squarefree)

<u>Conj.</u> It's $\prod_p \left(1 - \frac{c_p}{p^2}\right)$ where $c_p := \{n \in \mathbb{Z}/p^2\mathbb{Z} : n^4+1 = 0 \text{ in } \mathbb{Z}/p^2\mathbb{Z}\}$

<u>Proof?</u>

① Small primes

$\text{Prob}\left(p^2 \nmid n^4+1 \text{ for all } p \le r\right) = \prod_{p \le r} \left(1 - \frac{c_p}{p^2}\right).$

③ Large primes

$\#\{n \le B : p^2 \mid n^4+1 \text{ for some } p > B^2\} = 0$

② Medium-sized primes

For each $p \nmid \mathrm{disc}(x^4+1)$

$$\#\{n \le B : p^2 \mid n^4+1\} \text{ ~~for some p~~}$$

$$\le 4 \left\lceil \frac{B}{p^2} \right\rceil$$

$$\#\left\{ n \le B : \begin{array}{l} n \text{ is div. by } p^2 \\ \text{for } \underline{\text{some}}\ p \in (r, B^2) \end{array} \right\} \le \sum_{p \in (r, B^2)} 4 \left\lceil \frac{B}{p^2} \right\rceil$$

$$\le \sum_{p \in (r, B^2)} \left(4 \frac{B}{p^2} + 4 \right)$$

Browkin, Filaseta, Greaves, Schinzel

$$\le \frac{4}{r} B + 4 \frac{B^2}{\log B^2}$$

Granville

ABC conj. $\Longrightarrow$ Conj.

Closed points

X finite-type k-scheme
$\uparrow$ field

closed point P on X $\longleftrightarrow$ max. ideal $\mathfrak{m} \leq A$ for some affine open $\operatorname{Spec} A \hookrightarrow X$

residue field $\kappa(P) = A/\mathfrak{m}$

$\nwarrow$ finite ext. of k

make sense also for X f.type over $\mathbb{Z}$

$$\deg P := [A/\mathfrak{m} : k]$$

Ex. closed point on $\mathbb{A}^1$ $\longleftrightarrow$ max. ideal of $k[t]$
$\longleftrightarrow$ monic irred poly in $k[t]$
$\longleftrightarrow$ $\operatorname{Gal}(\bar{k}/k)$-orbit in $\mathbb{A}^1(\bar{k})$

More generally
closed pt. of X $\longleftrightarrow$ $\operatorname{Gal}(\bar{k}/k)$-orbit in $X(\bar{k})$

Zeta functions

$$\zeta_{\operatorname{Spec}\mathbb{Z}}(s) := \prod_{\text{prime } p} (1-p^{-s})^{-1} = \prod_{\substack{\text{closed pts. } P \\ \text{of } \operatorname{Spec}\mathbb{Z}}} \left(1 - \#\kappa(P)^{-s}\right)^{-1}$$

for $\operatorname{Re}(s) > 1$

Def. X f.type $/\mathbb{Z}$

$$\zeta_X(s) := \prod_{\text{closed } P \in X} \left(1 - \#\kappa(P)^{-s}\right)^{-1}$$

for $\operatorname{Re}(s) > \dim X$

Special case: X f.type $/\mathbb{F}_q$ Then $\#\kappa(P) = q^{\deg P}$

$$\zeta_X(s) = \prod_{\text{closed } P \in X} \left(1 - q^{-s \deg P}\right)^{-1}$$

$\Big\downarrow T = q^{-s}$

$$\rotatebox{90}{=}$$

$$Z_X(T) = \exp \sum_{r \geq 1} \frac{\#X(\mathbb{F}_{q^r})}{r} T^r$$

Bertini smoothness theorem

$X \subset \mathbb{P}^n$ over a field k

smooth of dim m
quasi-proj. subscheme

Then $\exists$ dense open $U \subseteq \check{\mathbb{P}}^n$ such that each $u \in U$ correspond to a hyperplane $H \subset \mathbb{P}^n_{k(u)}$ with $H \cap X$ smooth of dim $m-1$ over $k(u)$.

<u>Cor.</u> If k is infinite, $\exists\ H/_k$ s.t. $H \cap X$ is smooth.

Bertini smoothness theorem $/\mathbb{F}_q$

$S = \mathbb{F}_q[x_0, \dots, x_n]$ $\quad \mathbb{P}^n = \operatorname{Proj} S$

$S_d = \{\text{homog. polys. of deg } d \text{ in } S\}$ $\quad S_{\text{homog}} = \bigcup_{d \geq 0} S_d.$

If $f \in S_d$, $\quad H_f = \operatorname{Proj} S/(f)$

the hypersurface $f = 0$

For $\mathcal{P} \subseteq S_{\text{homog}}$

$$\mu(\mathcal{P}) := \lim_{d \to \infty} \frac{\# \mathcal{P} \cap S_d}{\# S_d}.$$

density

Thm. $X \subset \mathbb{P}^n$ over $\mathbb{F}_q$

smooth of dim m

quasi-proj.

$$\mathcal{P} = \left\{ f \in S_{\text{homog}} : \begin{array}{l} H_f \cap X \text{ is smooth} \\ \text{of dim } m-1 \end{array} \right\}$$

Then
$$\mu(\mathcal{P}) = \zeta_X(m+1)^{-1}$$
$$\in \mathbb{Q} \cap [0,1].$$

Comparison:

$$\text{Prob}\left(n=0 \text{ in } \operatorname{Spec}\mathbb{Z} \text{ is regular} \right) = \zeta(\underset{\uparrow}{2})^{-1}$$

dim Spec $\mathbb{Z}$ + 1

Example: $X = \mathbb{P}^2 \subset \mathbb{P}^2$ over $\mathbb{F}_2$

$$\# \mathbb{P}^2(\mathbb{F}_{2^r}) = 4^r + 2^r + 1 \rightsquigarrow Z_X(T) = \frac{1}{(1-T)(1-2T)(1-4T)}$$

Thm. $\Rightarrow$ $\text{Prob}\left(\begin{matrix}\text{plane curve in } \mathbb{P}^2 \text{ over } \mathbb{F}_2 \\ \text{is smooth}\end{matrix}\right) = \dfrac{21}{64}.$