Geometric Analytic # thy that will not appear ....

- Geometric twin primes / prime clusters

moduli space of $\underline{\text{factored}}$ f and $\underline{\text{factored}}$ g s.t. $g = f + 1$

$S_n \times S_n \circlearrowright (z_1, \dots, z_n, w_1, \dots, w_n)$ s.t.

$(x - z_1) \cdots (x - z_n) + 1$

$(x - w_1) \cdots (x - w_n)$

Pollack
Barry-Soroker

Geometric Linnik/Malle/Bhargava

X_n = moduli space of degree-d covers (or G-covers) of $\mathbb{P}^1$ with n branch points

(Hurwitz spaces)

Geometric Poonen-Rains

moduli space of elliptic surfaces together with elements of $H^2(\mathcal{E}, \mathbb{Z}/p\mathbb{Z})$

orthogonal pairing

Geometric Linnik-Duke

(Shankar-Tsimerman)

(generalized theta divisors inside abelian varieties)

Geometric Batyrev-Manin

$X_{\mathbb{F}_q}$ $X(\mathbb{F}_q(t))$ ~ maps from $\mathbb{P}^1$ to X

$X_n = \{\text{degree-}n \text{ maps from } \mathbb{P}^1 \text{ to } X\}$

Geometric Cohen-Lenstra

<u>Conj</u> Let ℓ an odd prime and $E_{r,\ell,N}$ be the expected value, as d ranges over sq. free integers in $[N,2N]$ of

$$\mathrm{Surj}(\mathrm{Cl}(\mathbb{Q}(\sqrt{d})), (\mathbb{Z}/\ell\mathbb{Z})^r)$$

$$\mathrm{Cl}(\mathbb{Q}(\sqrt{d})), (\mathbb{Z}/\ell\mathbb{Z})^r$$

e.g.

$E_{N,\ell,1}$ = expected value of

$$|C1[\ell]| - 1$$

Conjecture is that

$$\lim_{N \to \infty} E_{N,\ell,r} = 1$$

Function Field C-L

$\mathbb{Q}(\sqrt{-d})$

$\mathcal{O} \subset \mathbb{Q}(\sqrt{-d})$

ideal of $\mathcal{O}$

class gp of $\mathcal{O}$

$Cl(\mathcal{O})[\ell]$

$\ell \neq \text{char } \mathbb{F}_q$

n odd

$C_f : y^2 = f(x)$

$U = C_f - \infty_f$

effective divisor on U

$\text{Pic}(U)$

$\cong$

$\text{Jac}(C_f)(\mathbb{F}_q)$

$\text{Jac}(C_f)[\ell](\mathbb{F}_q)$

This suggests a definition:

$\mathrm{Conf}^n(\ell)$ is the moduli space of hyperelliptic curves with ℓ-level structure, i.e. pairs (f, P)

f sqfree monic degree n

$P \in \mathrm{Jac}(C_f)[\ell]$ <u>nonzero</u>.

$$\begin{array}{ccc} \mathrm{Conf}^n(\ell) & \xrightarrow{\pi} & \mathrm{Conf}^n \\ (f,P) & \longmapsto & f \end{array}$$

finite étale map of degree $\ell^{2g} - 1$

$$E_{q,\ell,r,n} =$$

$$\mathbb{E}_f \left| \mathrm{Surj}\left(\mathrm{Jac}(C_f)(\mathbb{F}_q), (\mathbb{Z}/\ell\mathbb{Z})^r\right) \right|$$

$E_{q,\ell,1,n} =$

averge of $|\mathrm{Jac}(C_f)(\mathbb{F}_q)[\ell]| - 1$

$=$ averge of $|\pi^{-1}(f)(\mathbb{F}_q)|$

$$= \frac{|\mathrm{Conf}^n(\ell)(\mathbb{F}_q)|}{|\mathrm{Conf}^n(\mathbb{F}_q)|}$$

So C-L predicts:

$$\lim_{n\to\infty} \frac{|\mathrm{Conf}^n(\ell)(\mathbb{F}_q)|}{|\mathrm{Conf}^n(\mathbb{F}_q)|} = 1$$

Note:

$$\lim_{q\to\infty} \frac{|\mathrm{Conf}^n(\ell)(\mathbb{F}_q)|}{|\mathrm{Conf}^n(\mathbb{F}_q)|} = \frac{q^n}{q^n} = 1$$

<u>IF</u> $\mathrm{Conf}^n(\ell)$ is geometrically irreducible (ie. connected)

Think of $\mathrm{Conf}^n(\mathbb{C}) \xrightarrow{\pi} \mathrm{Conf}^n$

as an action of

$$\pi_1(\mathrm{Conf}^n, f) \circlearrowright \pi^{-1}(f) \simeq (\mathbb{Z}/\ell\mathbb{Z})^{2g} - 0$$

$$\|$$

$$\mathrm{Br}_n$$

This is a linear action which preserves a symplectic form ie a map

$$\mathrm{Br}_n \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$$

$\mathrm{Conf}^n(\mathbb{C})$ is connected

$$\Updownarrow$$

Br_n acts <u>transitively</u> on nonzero vectors in $(\mathbb{Z}/\ell\mathbb{Z})^{2g}$

This is true, because we know that Br_n <u>surjects</u> onto $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$.

(A'Campo, Yu, Achter-Pries, Hall)

In fact, this big monodromy result gives

$$\lim_{q \to \infty} \text{average of } \mathrm{Surj}(\mathrm{Jac}(C_q)(\mathbb{Z}/\ell\mathbb{Z}))$$

$$= 1$$

for <u>all</u> r.

Amounts to studying action of $Sp_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ on $\mathrm{Surj}((\mathbb{Z}/\ell\mathbb{Z})^{2g}, \mathbb{Z}/\ell\mathbb{Z})$

On duality, on

$$\mathrm{Inj}\left((\mathbb{Z}/\ell\mathbb{Z})^2, (\mathbb{Z}/\ell\mathbb{Z})^{2g}\right)$$

But this action is <u>not</u> transitive!

e.g. given an injection through ℓth of irred. cpts

$$\Phi : (\mathbb{Z}/\ell\mathbb{Z})^2 \to (\mathbb{Z}/\ell\mathbb{Z})^{2g}$$

$$\Phi^*\omega$$

What's going on?

Really, we are thinking not about $Sp_{2g}(\mathbb{Z}/\ell\mathbb{Z})$

but $Sp_{2g}^{(q)}(\mathbb{Z}/\ell\mathbb{Z}) \subset GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z})$

It turns out that the action of Frob on these components is to multiply

$\Phi^*\omega$ by q.

The only one that's fixed is $\Phi^*\omega = 0$

i.e. the orbit of isotropic subspaces
of $(\mathbb{Z}/\ell\mathbb{Z})^{2g}$

UNLESS...

$q \equiv 1 \pmod{\ell}$

in which case all orbits are fixed,
and C-L has to change!

(see work of Garton)