- Counting squarefree integers in $[N, 2N]$

- Counting monic sq. free polynomials of degree n in $\mathbb{F}_q[t]$

- Computing cohomology of moduli space of deg-n sq frees in $\mathbb{C}[t]$

GEOMETRIC ANALYTIC #THY

- Start w/ problem over $\mathbb{Z}$

- Consider analogous problem over $\mathbb{F}_q(z)$: interpret as problem of studying $|X_n(\mathbb{F}_q)|$

$X_1, X_2, X_3, \dots$

- Formulate a geometric/topological assertion about $X_1, X_2, \dots X_n / \mathbb{C}$ which implies

Chowla conjecture:

$$\mu(n) = \begin{cases} 0 & n \text{ not squarefree} \\ (-1)^k & n = \prod k \text{ distinct primes} \end{cases}$$

This should act like a random sign (from an additive POV)

e.g. $\displaystyle\sum_{i=N}^{2N} \mu(n) = o(N)$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=N}^{2N} \mu(n) = 0$$

What about

$$\frac{1}{N}\sum_{N}^{2N}\mu(n)\mu(n+1)$$

Also $\to 0$?

More generally: **CONJ** (Chowla)

$$\sum_{n=N}^{2N}\mu(n+a_1)^{\varepsilon_1}\cdots\mu(n+a_r)^{\varepsilon_r}$$

$$= o(N)$$

Facts about arithmetic statistics in function fields in the "large q limit"

correspond to

Facts about irreducible components (i.e H^0) of moduli spaces.

$$\sum_{f \text{ of degree } n} \mu(f)\mu(f+1)$$

$\Delta(f) = 0 \Longleftrightarrow f$ not sq free

$\Updownarrow$

$\mu(f) = 0$

In fact,

$$\mu(f) = (-1)^n \chi(\Delta(f))$$

where $\chi : \mathbb{F}_q^* \to \pm 1$

So $\mu(f)\mu(f+1) =$

$$\chi(\Delta(f))\chi(\Delta(f+1))$$

$$= \chi(\Delta(f)\Delta(f+1))$$

$=$ # square roots of $\Delta(f)\Delta(f+1)$

-1

So let Y_n be the moduli space of pairs (f, y) where y is a square root of $\Delta(f)\Delta(f+1)$

i.e. Y_n has equation

$$y^2 = \Delta(f)\Delta(f+1)$$

$$\begin{aligned} Y_n &\to \mathbb{A}^2 \\ (f,y) &\mapsto f \end{aligned}$$ is a double cover ramified at $V(\Delta(f)\Delta(f+1))$

So $\sum_{f} \mu(f)\mu(f+1)$

$$= \sum_{f} \left(\begin{array}{c} \#\text{ square roots of } \Delta(f)\Delta(f+1) \\ -1 \end{array} \right)$$

$$= |Y_n(\mathbb{F}_q)| - q^n$$

We hope this is $o(q^n)$

ie we want

$$|Y_n(\mathbb{F}_q)| = q^n + o(q^n)$$

One might conjecture

(Geometric Chowla)

For all $n \geq 0$, Y_n is irreducible,

and there is a constant $\alpha > 0$

s.t. $H^{2n-i}_{et;c}(Y_n; \mathbb{Q}_\ell) = 0$

for all $i < \alpha n$

Instead of asking about

$$\lim_{n\to\infty} q^{-n} |Y_n(\mathbb{F}_q)|$$

What about

$$\lim_{q\to\infty} q^{-n} |Y_n(\mathbb{F}_q)|$$

or

$$\lim_{n\to\infty} \lim_{q\to\infty} q^{-n} |Y_n(\mathbb{F}_q)|$$

Thm (Carmon-Rudnick)

Let $\mathbb{F}_q$ be odd characteristic and let $a_1, \dots, a_m$ be distinct polynomials in $\mathbb{F}_q[t]$ degree $< n$.

Then
$$\sum_{\deg f = n} \mu(f+a_1)^{\varepsilon_1} \cdots \mu(f+a_m)^{\varepsilon_m}$$
(not all exponents even)
$$\le 2mn q^{n-1/2} + 3mn^2 q^{n-1}$$
$$= o(q^n) \text{ as } q \to \infty \text{ with } n \text{ fixed.}$$

Main idea:

$$|Y_n(\mathbb{F}_q)| =$$

$$\sum (-1)^i \operatorname{Tr} \operatorname{Frob} H^{2n-i}_{et,c}(Y_n; \mathbb{Q}_\ell)$$

Weil bounds (Deligne) give upper bounds on eigenvalues of Frobenius acting on $H^{2n-i}_{et,c}(Y_n; \mathbb{Q}_\ell)$

e-vals have $|\ |$ at most $q^{n-i/2}$

total of all Betti numbers can be bounded independent of q by B_i

so contribution of H^{2n-i} for all $i>0$ is at most

$$Bq^{n-1/2}$$

while $H^{2n}(Y_n; \mathbb{Q}_\ell)$ contributes

''''

$H^{2n}_{\acute{e}t,c}(Y_n, \mathbb{Q}_\ell) =$

$\mathbb{Q}_\ell$-v.s. spaned by irreducible components of Y_n

Frob by q^n. permutation action on components.

$\mathbb{A}^1 \sqcup \mathbb{A}^1(\mathbb{F}_q) = 2q$

$\mathrm{Tr}\ \mathrm{Frob}\ H^{2n} = \#\ \mathbb{F}_q$-rat'l irred. cpts.

So one needs to show

Y_n is geometrically irreducible

this is true unless

$$\Delta(f)\Delta(f+1)$$

is a perfect square

It's not :D

One way to think of this:
can think of the étale double
cover of $\mathbb{A}^n - V(\Delta(f)\Delta(f+1))$
given by adjoining $\sqrt{\Delta(f)\Delta(f+1)}$
(this is open in Y_n)
This is given by a map
$$\mathrm{Gal}(k(a_1,\ldots,a_n)) \to \mathbb{Z}/2\mathbb{Z}$$
We need to know it ~~is surjective~~ $\boxed{\text{BIG MONODROMY}}$