$sf_q(n)$ = # monic sqfree polys $\in \mathbb{F}_q[t]$ of degree n

$$\lim_{n\to\infty} q^{-n} sf_q(n) = 1 - 1/q$$

In fact, ~~fo~~

$$sf_q(n) = q^n\left(1 - \frac{1}{q}\right)$$

for all $n \geq 1$

Pf Let $\Sigma_{n,e}$

= monic poly. deg n

of the form $a(t)\, b^2(t)$

a sqfree b degree e.

all q^n polys can be factored (uniquely) in this way,

$$q^n = \sum_{e=0}^{\lfloor n/2 \rfloor \, \infty} |\Sigma_{n,e}|$$

$$|\Sigma_{n,e}| = q^e \, sf_q(n-2e)$$

by induction, starting from $sf_q(0) = 1$

$sf_q(1) = q$

Remark: the absence of an error term is misleading: for a general function field $\mathbb{F}_q(C)$ the natural analogue

$$sf_C(n) = \zeta_C(2)^{-1} \cdot q^n + \text{error term}$$

(Byungchul Cha, 2011)

How do you tell whether an integer / polynomial is squarefree?

$\mathbb{Z}$ - somewhat hard

$k[t]$ - compute <u>discriminant</u>

$$P(t) = t^3 + a_1 t^2 + a_2 t + a_3$$

is squarefree

$$\Longleftrightarrow$$

$$a_2^2 a_1^2 - 4a_3 a_1^3 - 4a_2^3 + 18 a_3 a_2 a_1$$

$$-27 a_3^2 \neq 0$$

θ_i roots of P

In fact, this is

$$\Delta(P) = \prod_{i \neq j} (\theta_i - \theta_j)$$

Since this is preserved by S_n action on roots, it is a polynomial in the a_i.

THE MODULI SPACE OF SQUAREFREE POLYNOMIALS

open subvariety of $\mathbb{A}^n_{a_1,\dots,a_n}$ $\leftarrow$ moduli space of monic degree n polynomials

where Δ does not vanish.

Denote this space Conf^n

$\mathrm{Conf}^n(k) =$ sqfree monic polys degree n in $k[z]$

$sf_q(n) = |\mathrm{Conf}^n(\mathbb{F}_q)|$

$\mathrm{Conf}^n(\mathbb{C}) = \{$ <u>unordered</u> n-tuples of <u>distinct</u> complex numbers $\}$

$$P \longmapsto \{\text{roots of } P\}$$

$\mathrm{Conf}^n =$ configuration space of n pts on $\mathbb{C}$.

$\mathrm{Conf}^n(\mathbb{C})$ is a <u>manifold</u> - what can we say about its geometry?

$\mathrm{Conf}^1(\mathbb{C}) = \mathbb{C}$

$\mathrm{Conf}^2(\mathbb{C})$ is a circle

$\uparrow$

<u>ordered</u> pairs of distinct pts

$\cdot z_2$

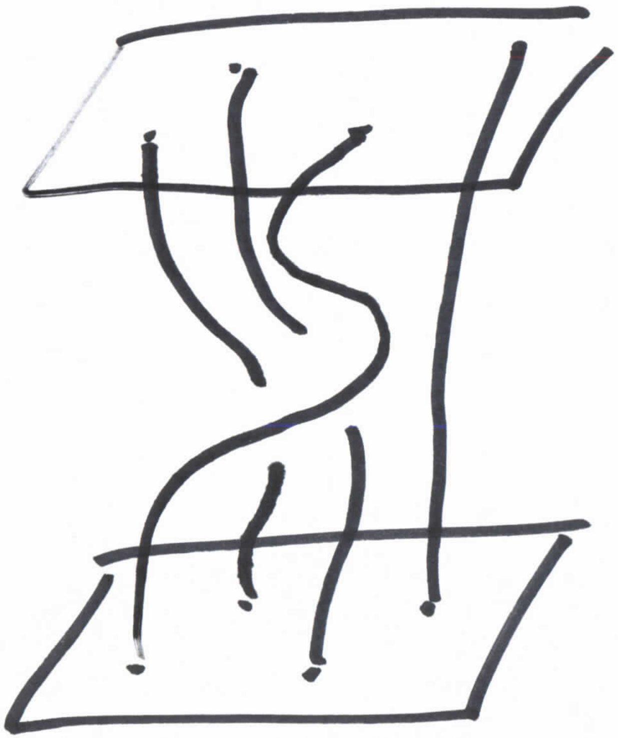
$\cdot z_1$

$z_1 \in \mathbb{C}$

$z_2 \in \mathbb{C} - z_1$

As n grows, Conf^n gets more complicated

e.g. the fundamental gp $\pi_1(\mathrm{Conf}^n)$



$\pi_1(\mathrm{Conf}^n) = \mathrm{Br}_n$

<u>THM</u> (Arnold) For all $n > 1$

$$H^0(\mathrm{Conf}^n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}$$
$$H^1(\mathrm{Conf}^n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}$$
$$H^i(\mathrm{Conf}^n(\mathbb{C}), \mathbb{Q}) = 0 \quad i > 1$$

example of <u>stable cohomology</u> –

(see e.g. $\mathcal{M}_g$

étale cohomology story

Grothendieck-Lefschetz trace formula

$$|X(\mathbb{F}_q)| = \sum_i (-1)^i \operatorname{Tr} \mathrm{Frob} \,|\, H^i_{et,c}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$$

$$\begin{array}{c} \circlearrowright \\ \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \\ \parallel \\ \langle \mathrm{Frob} \rangle \end{array}$$

When X is a variety that "makes sense" over both $\mathbb{C}$ and $\mathbb{F}_q$

$(X \to \mathrm{Spec}\,\mathbb{Z})$

we may hope

$$\dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(X_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$$

$$\dim_{\mathbb{Q}} H^i(X(\mathbb{C}); \mathbb{Q})$$

under good circumstances this is so

In the case of Conf^n, we indeed have

<u>FACT</u>: For all $n \geq 2$,

$$H^0_{et}(\mathrm{Conf}^n / \overline{\mathbb{F}}_q ; \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

$$H^1(\qquad\qquad) = \mathbb{Q}_\ell$$

$$H^i(\qquad\qquad) = 0 \quad \forall i > 1$$

Moreover, Frob acts as 1 on H^0_{et} and as q on the H^1_{et}.

Poincaré duality relates $H^i_{ét}$ with $H^{2n-i}_{ét;c}$

$$|\mathrm{Conf}^n(\mathbb{F}_q)| = q^n \cdot \sum (-1)^i \,\mathrm{Tr}\,\mathrm{Frob}\, H^i_{ét}(\overset{\mathrm{Conf}^n}{\cancel{X}}/\overline{\mathbb{F}}_q, \mathbb{Q}_\ell)^\vee$$

$$= q^n(\underbrace{\mathrm{Tr}\,\mathrm{Frob}|H^{0\vee}}_{1} - \underbrace{\mathrm{Tr}\,\mathrm{Frob}|H^{1\vee}}_{1/q})$$

$$= q^n - q^{n-1}$$