ANALYTIC NUMBER THY?

- How many pairs of coprime integers in

$$[1, N] \times [1, N]$$

- If X a proj. variety, how many pts in $X(\mathbb{Q})$ of height at most N?

- How many primes $\leq N$?
- How many totally real cubic fields with discriminant $\leq N$?
- How many totally real cubic fields with <u>prime</u> discriminant $\leq N$?

- Autocorrelation of Möbius (CHOWLA)

$$\sum_{n \leq N} \mu(n)\mu(n+1) = o(N)$$

- What is the probability that a quadratic imaginary field $\mathbb{Q}(\sqrt{-d})$ (d random $[N, 2N]$) has class number prime to 7?
(Cohen-Lenstra)

If n is a random squarefree in $[N, 2N]$, what is the probability that $\exists$ a totally real quintic field $K/\mathbb{Q}$ with discriminant n?

$$\left(e^{-1/120}\right)$$

"How many" is meant <u>asymptotically</u>

$$\left|\{(x,y) \in [1,N] \times [1,N] \text{ coprime}\}\right| = \frac{6}{\pi^2} N^2$$

NO

$$\lim_{N \to \infty} N^{-2} \left|\{(x,y) \text{ coprime in box}\}\right| = \frac{6}{\pi^2}$$

Or more:

$$\left| \begin{array}{c} (x,y) \text{ coprime in} \\ [1,N] \times [1,N] \end{array} \right|$$

$$= \frac{6}{\pi^2} N^2 + O(N^{2-\delta})$$

for $\delta > 0$ (power-saving error term)

Mostly we will consider just two fields: $\mathbb{Q}$ and $\mathbb{F}_q(t)$

$$\mathbb{Z} \subset \mathbb{Q}$$

$= x \in \mathbb{Q}$: $|x|_p \leq 1$ for all absolute nonarch. values $|\ |_p$ except $|\ |_\infty$.

Analogously:

$$\mathbb{F}_q[t] \subset \mathbb{F}_q(t)$$

$$= x \in \mathbb{F}_q(t)$$

For each point P of $\mathbb{P}^1$,

$$|x|_P = q^{-\mathrm{ord}_P(x)}$$

$$x = \frac{P}{Q},$$

$$\mathrm{ord}_\infty(x) = \deg Q - \deg P$$

Analogous subring of $\mathbb{F}_q(t)$

is

$x : |x|_P \leq 1$ all P except ∞

i.e. x has no denominator

i.e. x has no poles away from ∞

i.e. x is a polynomial P

$$|x|_\infty = q^{\deg P}$$

A difference:

in $\mathbb{Q}$, ∞ is special

(the only archimedean place)

in $\mathbb{F}_q(t)$, ∞ is not special –

we can apply an automorphism of $\mathbb{P}^1$ to move it around, e.g.

$\mathbb{F}_q\left[\frac{1}{1-t}\right]$

positive integers -
coset representatives
for $\mathbb{Z}/\mathbb{Z}^*$

monic polynomials
coset rep'tives for
$\mathbb{F}_q[t]/(\mathbb{F}_q[t])^*$

an interval in $\mathbb{Z}$ is

$$n : |n - n_0| \leq d$$

an interval in $\mathbb{F}_q[t]$ is

$$f : |f - f_0| \leq e$$

$$\begin{array}{c} || \\ q^{\deg(f - f_0)} \end{array}$$

e.g.

$$f : |f - x^n| \leq q^{n-1}$$

= monic polynomials of degree n.

SQUAREFREE INTEGERS & SQUAREFREE POLYNOMIALS

Q: How many integer in $[N, 2N]$ are squarefree?

One might expect Prob(squarefree) to be

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right) \ldots$$

and indeed this is so:

if $sf(N) = \#$ sqfrees in $[N, 2N]$,

then

$$\lim_{N \to \infty} N^{-1} sf(N) = \prod_p (1 - p^{-2}) = \zeta(2)^{-1}$$

Over $\mathbb{F}_q[t]$,

our interval is monic polynomials of degree n

$$x^n + a_1 x^{n-1} + \dots + a_n$$

an interval of size q^n

(So think of q^n as N)

$sf_q(n) =$ # monic squarefree polys of degree n

~~What is~~

$$\lim_{n\to\infty} q^{-n} sf_q(n) = 1 - \frac{1}{q}$$

Heuristically, one might expect

$$\lim_{n\to\infty} q^{-n} sf_q(n) = \prod_{\substack{P \\ \text{irreducible} \\ \text{monic}}} \left(1 - q^{-2\deg P}\right)$$

$$= \prod_p \left(1 - |p|^{-2}\right)$$

$$= \zeta_{\mathbb{F}_q[T]}(2)^{-1}$$

$$\overset{!!}{\underset{!!}{=}} \left(1 - \frac{1}{q}\right)$$