CURVES AND ZETA FUNCTIONS OVER FINITE FIELDS
POSSIBLE PROJECTS

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For background material, and more details about the projects, see the notes.

1. One-level density for cyclic trigonal curves

The first project concerns the computation of the average number of points of $\mathbb{F}_{q^n}$, the one-level density and the n-level density for the family of cyclic trigonal curves, which is a family with unitary symmetries. We are interested in those statistics for q fixed. When $q \to \infty$, the statistics are given by statistics on spaces of random matrices.

For the case of hyperelliptic curves, those statistics are computed in [Rud10, RG12, ERGR13]. Some interesting point about those statistics for hyperelliptic curves is that at the g-limit, the average number of points over $\mathbb{F}_{q^n}$ agrees with the random matrix model $\mathrm{USp}(2g)$ for $n \in \left(3\log_q(g), 4g - 5\log_q(g)\right), n \neq 2g$, and there is a deviation from the random matrix model for small n, and for $n = 2g$, and this introduces a lower order term in the one-level density. It is believed that this lower order term should not be present in the number field case, but interestingly, this is still unknown, as the lower order terms can be controlled only for test functions with Fourier transform of small support in the case of quadratic Dirichlet L-functions.

Let $\mathcal{H}^{(d_1,d_2)}$ be an irreducible component of the moduli space of cyclic trigonal curves of genus $g = d_1 + d_2 - 2$. For each curve $C \in \mathcal{H}^{(d_1,d_2)}$, let Θ_C be the $2g \times 2g$ matrix with eigenvalues $e^{i\theta_j}$, $j = 1, \dots, 2g$. At the q-limit, we have the following result.

Theorem 1.1. *Let d_1, d_2 be fixed. Then, as $q \to \infty$ and C runs over the moduli space $\mathcal{H}^{(d_1,d_2)}$ of cyclic trigonal curves of genus $g = 3(d_1 + d_2 - 1)/2$, the matrices Θ_C become equidistributed with respect to the Haar measure on $U(2g)$, i.e.*

$$\lim_{q\to\infty} \langle \operatorname{tr} \Theta_C^n \rangle = \int_{U(2g)} \operatorname{tr} U^n \, dU.$$

Then, for g fixed and $q \to \infty$, it follows that

$$\langle \operatorname{tr} \Theta_C^n \rangle = \int_{U(2g)} \operatorname{tr} U^n \, dU = 0, \quad n \in \mathbb{Z}, n \neq 0. \tag{1}$$

More generally, we have the following result for the average product of powers of traces for eigenvalues of random matrices in $U(N)$, which give all statistics on $\mathcal{H}^{(d_1,d_2)}$ at the q-limit.

Theorem 1.2. [DS94] *Let $r_1, \dots, r_n$ be non-zero integers $\sum_{i=1}^n |r_i| \le 2N$. Let $s_1, \dots, s_m$ be the distinct values appearing in the list $|r_i|$, $i = 1, \dots, n$, and let a_j (respectively b_j) be the number of times each value s_j (respectively $-s_j$) occurs. Then,*

$$M(r_1, \dots, r_n; N) \quad := \quad \int_{U(N)} \prod_{i=1}^n \operatorname{tr} U^{n_i} \, dU = \begin{cases} \prod_{j=1}^m a_j! s_j^{a_j}, & \text{if } a_j = b_j, j = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases}$$

For finite q, we study the average $\langle \operatorname{tr} \Theta_C^n \rangle$ by using the explicit formula

$$-q^{n/2} \sum_{i=1}^{2g_F} e(n\theta_{F,i}) = \sum_{\deg P | n} \deg P \chi_F(P)^{n/\deg P} + \overline{\chi_F}(P)^{n/\deg P} + \lambda_F,$$

and we have to compute the characters sums as in [Rud10], but for cubic characters. Some computations are done in [Xio10], but summing over all $n \le K$ (using the Beurling-Selberg polynomials of level K).

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2. One-level density for general trigonal curves

The moments of traces (or equivalently the average number of points of $\mathbb{F}_{q^n}$), the one-level density and more generally the n-level density can also be studied for the family of cyclic trigonal curves. This is a family with symplectic symmetry (for evidence for the number field case, see the work of Yang [Yan09]). When $q \to \infty$, the statistics are given by statistics on $\mathrm{USp}(2g)$, as the family of hyperelliptic curves (see section 6.2 of the notes for details about the distribution when $q \to \infty$, and for q fixed). Again, we expect the statistics to agree with the statistics from the random model even without taking the q-limit, except from possible deviation for small n, and possibly $n = 2g$. In particular, since the family of general trigonal curve is a symplectic family, we expect to see the "bias" in the number of points over extensions $\mathbb{F}_{q^n}$ when n is even.

Some estimates on the higher moments of traces $\operatorname{tr} \Theta_C^n$ (which are the needed for the average number of points over $\mathbb{F}_{q^n}$ and the one-level density as explained in the notes) can be found in [TX14].

3. Average number of points over $\mathbb{F}_{q^n}$ as a sum of independent random variables

The distribution of the number of points over $\mathbb{F}_{q^n}$ can also be expressed as a sum of random variables (as done in Section 6 of the notes for $n = 1$). We expect that the variables will be independent, but not identically distributed, as primes of different degrees (or points defined over different extensions of $\mathbb{F}_q$) will give raise to different variables. Two techniques can be used: counting the number of points on curves for values of $x \in \mathbb{P}^1(\mathbb{F}_{q^n})$ as it was done for hyperelliptic curves and cycle ℓ-covers in the notes, or using the densities of extensions with splitting conditions at different primes as for general trigonal curves.

4. Zeros distribution on curves with more points

The zeta zero statistics calculated in [FR10, Ent12, BDF$^+$12, BDFL] have in common the fact that the average number of points on the curves in the families considered is most of the time $q + 1$, the same as on $\mathbb{P}^1(\mathbb{F}_q)$. It would be interesting to see if one can extend the same techniques to compute zeta zeros statistics for families where the average number of points is greater. Examples of such families can be found in [Woo12, EW12]. This is related to Project 2 and Project 3.

5. Artin-Schreier meets hyperelliptic

For $p = 2$, Artin-Scheier curves become hyperelliptic curves in characteristic 2. The moduli space is better understood (see [PZ12]). It might be interesting to study points statistics and level one density in this case. All the work done so far Artin-Schreier curves and hyperelliptic curves was in odd characteristic. The first thing to look at is how all the basic Artin-Schreier theory changes (e.g. genus formula, character theory). Then one would have to go through [PZ12] to understand the stratification of the moduli space into p-rank strata. Even better would be to have the stratification in terms Newton polygons, but this might be out of reach with the current technology. Zhu's paper [Zhu06] might also provide some insight.

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