$C : y^3 = F_1 F_2^2 \qquad g = d_1 + d_2 - 2 \qquad d_1 + 2d_2 \equiv 0\ (3)$

$$\langle C(\mathbb{F}_q) \rangle_{\mathcal{H}(d_1,d_2)} = \Big\langle \sum_{a \in \mathbb{P}^1(\mathbb{F}_q)} 1 + \chi_3(F(a)) + \chi_3^2(F(a)) \Big\rangle_{\mathcal{H}(d_1,d_2)}$$

$$= q+1$$

In fact, the distribution of $\# C(\mathbb{F}_q)$ is the same as the distribution of $\sum_{i=1}^{q+1} X_i$

$$X_i = \begin{cases} 0 & \dfrac{2q}{3(q+2)} \\ 1 & \dfrac{2}{q+2} \\ 3 & \dfrac{q}{3(q+2)} \end{cases} \qquad d_1, d_2 \to \infty$$

<u>BDFL</u>

$$\mathcal{H}_{g,3} = \bigcup_{d_1+d_2-2=g} \mathcal{H}^{(d_1,d_2)}$$

David Y

BDFL + Kaplan-Ozman

work directly with $\frac{Z_K(u)}{Z_K(u)}$ (following Wood)

$$\langle C(\mathbb{F}_q)\rangle_{\mathcal{H}_{g,2}} = \sum_{\substack{P\in S_K \\ \deg P=1 \\ \cancel{P \text{ ramifies}}}} \frac{\#E_3(K,g,P,\mathrm{RAM})}{\#E_3(K,g)} \to \frac{2}{q+2}$$

$[\,K=\mathbb{F}_q(C),\ C: y^3 = F_1 F_2^2 \text{ of genus } g\,]$

$$+ \sum_{\substack{P\in S_K \\ \deg P=1}} 3\,\frac{\#E_3(K,g,P,\mathrm{SPLIT})}{\#E_3(K,g)} \to \frac{q}{3(q+2)}$$

Counts the ratio

$$\frac{\#E_3(K,g,\mathcal{P},\varepsilon)}{\#E_3(K,g)}$$

$\mathcal{P}=\{P_1,\dots,P_{q+1}\}$

we get the istribution

Distribution for $\mathcal{H}_{g,3}$ fits the RV of $\mathcal{H}^{(d_1,d_2)}$

$$\# \mathcal{H}_{g,3} = \sum_{\substack{d_1+d_2-2=g \\ d_1+2d_2 \equiv 0\,(3)}} \# \mathcal{H}^{(d_1,d_2)}$$

using CFT

BDFL

$d_1, d_2 \to \infty$

<u>Trigonal Curves</u> (Wood)

K/k ~~deg~~ cubic non Galois, $k = \mathbb{F}_q(C)$

Work directly with

$$\frac{Z_K(u)}{Z_k(u)} = P_C(u) = \prod_{j=1}^{2g} \left(1 - u\alpha_j(C)\right)$$

$$\frac{Z_K(u)}{Z_k(u)} = \prod_{\substack{p \in S_k \\ p \in S_{111}}} \frac{(1-u^{\deg p})^{-3}}{(1-u^{\deg p})^{-1}} \prod_{\substack{p \in S_3 \\ p \in S_k}} \frac{(1-u^{3\deg p})^{-1}}{(1-u^{\deg p})^{-1}}$$

$$\prod_{\substack{p \in S_k \\ p \in S_{11^2}}} \qquad \prod_{\substack{p \in S_k \\ p \in S_{12}}} \qquad \prod_{\substack{p \in S_k \\ p \in S_{1^3}}} \frac{(1-u^{\deg p})^{-1}}{(1-u^{\deg p})^{-1}}$$

<u>Explicit Formula</u>

$$\sum_{j=1}^{2g} \alpha_j(C)^n = \sum_{\substack{P \in \cancel{S_k}\, S_{111} \\ \deg P \mid n}} 2\deg P + \sum_{\substack{P \in S_{12} \\ \deg p \mid \frac{n}{2}}} 2\deg P$$

$$+ \sum_{\substack{P \in S_{11^2} \\ \deg p \mid n}} \deg p + \sum_{\substack{p \in S_3 \\ \deg \mid n/3}} 3\deg p - \sum_{\substack{p \in S_3 \\ \deg p \mid n}} \deg p$$

$\underline{n=1}$ $\quad E_3(k,g)$ = all cubic non Galois $/k$ of genus g

$$\langle C(\mathbb{F}_q) - (q+1) \rangle = 2 \sum_{\deg P = 1} \frac{\# E_3(k, g, P, 111)}{\# E_3(k,g)}$$

$$+ \sum_{\deg P = 1} \frac{\# E_3(k, g, P, 11^2)}{\# E_3(k,g)}$$

$$+ \sum_{\deg P = 1} \frac{\# E_3(k, g, P, 3)}{\# E_3(k,g)}$$

Those were counted by Datskovsky & Wright

$$= q + 2 - \frac{1}{q^2+q+1}$$

We also know the distribution by using

$$\frac{\# E_3(k, g, \mathcal{P}, \varepsilon)}{\# E_3(k,g)}$$

Then we saw that $\underbrace{\# C(\mathbb{F}_q)}_{C \in \mathcal{F}(g,q)}$ is distributed as

- $\mathrm{tr}(\Theta A)$ in some matrix space as $q \to \infty$

  hyperelliptic $USp(2g)$

  cyclic trigonal $U(2g)$

  cubic non Galois $USp(2g)$

- as a sum of iid when $g \to \infty$

What if $q, g \to \infty$? We can then show that $q^{-1/2}[\# C(\mathbb{F}_q) - q - 1]$ is a $N(0,1)$ as $q, g \to \infty$ by computing all moments hyperelliptic curves (Kurlberg-Rudnick)

cyclic trigonal (BDFL)

non Galois cubic (Thorne - Xiong).

Traces of high powers and one level density

$$\#C(\mathbb{F}_{q^n}) = q \sum_{j=1}^{2g} \alpha_j(C)^n = q^{n/2}\, \mathrm{tr}(\Theta_C^n)$$

<u>hyperelliptic</u> <u>curves</u> (Rudnick for q fixed, $g \to \infty$)

<u>Thm</u> (Katz Sarnak Equidistribution thm)

$$\lim_{q \to \infty} \langle \mathrm{tr}\, \Theta_C^n \rangle_{\mathcal{H}_g(\mathbb{F}_q)} = \int_{USp(2g)} \mathrm{tr}(U^n)\, dU$$

$$= \begin{cases} 2g & n = 0 \\ -\eta_n & 1 \leq n \leq 2g \\ 0 & n > 2g \end{cases}$$

$$\eta_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Now fix q & let $g \to \infty$

<u>Thm</u> $Y^2 = F(X)$

$$\langle \operatorname{tr} \Theta_F^{\,n} \rangle_{\mathcal{H}_g(\mathbb{F}_q)} = \begin{cases} -\eta_n & 1 \leq n < 2g \\ -1 - \dfrac{1}{q-1} & n = 2g \\ 0 & n > 2g \end{cases}$$

$$+ \eta_n \frac{1}{q^{n/2}} \sum_{\deg P \mid \frac{n}{2}} \frac{\deg P}{|P| + 1} + O_q\left(n q^{-n/2 - 2g}\right).$$

Cor For $3\log q < n < 4g - \log q$, we have

$$\langle \operatorname{tr} \Theta_f^n \rangle = \int_{USp(2g)} \operatorname{tr} U^n \, dU + \underline{o}\left(\frac{1}{g}\right)$$

Cor $\langle C(\mathbb{F}_{q^n}) \rangle_{\mathcal{H}_{g}(\mathbb{F}_q)}$ $= \begin{cases} q^n + q^{n/2} + \underline{o}(q^{n/2}) & n \text{ even} \\ q^n + \underline{o}(q^{n/2}) & n \text{ odd} \end{cases}$

$C(\mathbb{F}_{q^n}) \to C/\mathbb{F}_q$

for n in the range of Cor above.

All $\operatorname{tr}(\Theta_f^n)$ determine the one level density, and enough $\operatorname{tr}(\Theta_f^n)$ determine the one level density for $\operatorname{supp}(\hat{f}) \subseteq (-\delta, \delta)$

<u>Cor</u> Let $f \in S(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq (-2, 2)$,

then

$$\lim_{g \to \infty} \langle \underset{\substack{\uparrow \\ \text{one level density}}}{W_f(C)} \rangle = \lim_{g \to \infty} \int_{USp(2g)} W_f(U)\, dU$$

$$= \int_{-\infty}^{\infty} f(x) \left[1 - \frac{\sin(2\pi x)}{2\pi x} \right] dx$$

What about $\overset{\text{cyclic}}{\text{trigonal}}$ curves & cubic non Galois curves?

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$U(2g) \qquad\qquad\qquad\qquad USp(2g)$$

- Can you compute $\langle tr(\Theta_C^n) \rangle$
- Can you compute one level density for $\text{supp}(\hat{f}) \subseteq (-\delta, \delta)$
- Can you compute $\langle tr\,\Theta_C^{n_1} \cdots tr\,\Theta_C^{n_k} \rangle$ & then the k-level density for $\text{supp}(\hat{f}) \subseteq (-\delta, \delta)$