Thm (Weil) Let C be a smooth projective curve of genus g / $\mathbb{F}_q$ & $Z_C(u) = \exp\left(\sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n}) \frac{u^n}{n} \right)$

Then $Z_C(u) = \dfrac{P_C(u)}{(1-u)(1-qu)}$ where $P_C(u) \in \mathbb{Z}[u]$ of degree $2g$

and $P_C(u) = \prod_{j=1}^{2g} (1 - u\,\alpha_j(C))$

with $|\alpha_j(C)| = \sqrt{q}$

Then taking log~~arithmic derivative~~

$$\#C(\mathbb{F}_{q^n}) - (q^n + 1) = -\sum_{j=1}^{2g} \alpha_j(C)$$

so statistics on the zeroes give statistics on # points

$$|\#C(\mathbb{F}_{q^n}) - (q^n+1)| \leq 2g\sqrt{q}^{\,n}$$ Weil's bound.

Notation Θ_C = $2g \times 2g$ matrix with diagonal entries $e^{i\theta_j(C)}$, $\alpha_j(C) = \sqrt{q}\, e^{i\theta_j(C)}$

David 3

Want to study the distribution of the zeroes when C varies over a family of curves of genus g over $\mathbb{F}_q$ $\rightsquigarrow$ $\mathcal{F}(g,q)$

<u>Thm</u> (Deligne's Equidistribution thm)

Let $\mathcal{M}_g(\mathbb{F}_q)$ be the moduli space of curves of genus g over $\mathbb{F}_q$ & Let f be any class function on $USp(2g)$ (which is the monodromy group of the family)

Then

$$\lim_{q\to\infty} \frac{\sum'_{C\in\mathcal{M}_g(\mathbb{F}_q)} f(\Theta_C)}{\sum'_{C\in\mathcal{M}_g(\mathbb{F}_q)} 1} = \int_{USp(2g)} f(U)\,dU$$

<u>Remark</u> This is one of the key ingredient of Katz-Sarnak's proof of Montgomery's PCC over function fields

What about q fixed & $g \to \infty$. What statistics do you get? First work Kurlberg - Rudnick

1) Hyperelliptic curves

$\mathcal{H}_g(\mathbb{F}_q)$ = moduli space of hyperelliptic curves over $\mathbb{F}_q$.

$$Y^2 = F(x) \qquad F(x)\ SF, \quad g = \left\lfloor \frac{d-1}{2} \right\rfloor, \ d = \deg F$$

$$\lim_{g \to \infty} \frac{\#\{C \in \mathcal{H}_g(\mathbb{F}_q) : \#C(\mathbb{F}_q) = m\}}{\#\{C \in \mathcal{H}_g(\mathbb{F}_q)\}} \sim \text{Prob}\left(\sum_{i=1}^{q+1} X_i = m\right)$$

where the X_i are i.i.d. such that

$$X_i = \begin{cases} 0 & \text{with Prob } \frac{q}{2(q+1)} \\ 1 & \text{with Prob } \frac{1}{q+1} \\ 2 & \text{with Prob } \frac{q}{2(q+1)} \end{cases}$$

At the end, this depends on

$$\operatorname*{Prob}_{a \in \mathbb{F}_q}(F(a)=0) = \frac{\#\{F \in \mathcal{F}_d : F(a)=0\}}{\#\mathcal{F}_d} = \frac{1}{q+1}$$

$\mathcal{F}_d$ = SF poly of degree d
(monic)

<u>Remark</u> $\frac{q-1}{q^2-1} = \frac{1}{q+1}$

$$\langle \# C(\mathbb{F}_q) \rangle = \langle \sum_{a \in \mathbb{P}^1(\mathbb{F}_q)} 1 + \chi_2(F(a)) \rangle$$

$$= \langle q+1 + \sum_{a \in \mathbb{P}^1(\mathbb{F}_q)} \chi_2(F(a)) \rangle$$

<u>Thm</u>

$$\frac{\#\{F \in \mathcal{F}_d : F(a_i) = \alpha_i \quad 1 \le i \le q\}}{\#\mathcal{F}_d} \sim \left(\frac{1}{q+1}\right)^m \left(\frac{q}{(q+1)(q-1)}\right)^{q-m}$$

$(\mathbb{F}_q) = \{a_1, \dots, a_q\}$

$m = \#$ of α_i which are 0

2) $Y^3 = F(x)$ cyclic trigonal curves $(q \equiv 1 (3))$

$F(x)$ cubefree

$Y^3 = F_1(x) F_2^2(x)$ $\quad (F_1, F_2) = 1$, F_1, F_2 SF

$\deg F_i = d_i$

$g = d_1 + d_2 - 2 \quad (d_1 + 2d_2 \equiv 0 (3))$

$$\mathcal{H}_{g,3} = \bigcup_{\substack{d_1 + 2d_2 \equiv 0 (3) \\ g = d_1 + d_2 - 2}} \mathcal{H}^{(d_1, d_2)}$$

$$\# C(\mathbb{F}_q) = \sum_{a \in \mathbb{P}^1(\mathbb{F}_q)} 1 + \chi_3(F(a)) + \chi_3^2(F(a))$$

where χ_3 is the cubic residue symbol $/\mathbb{F}_q$

$$\mathcal{F}_{(d_1, d_2)} = \{ F_1, F_2 : (F_1, F_2), SF, \deg F_i = d_i \}.$$

$$\# \mathcal{F}_{(d_1, d_2)} \sim \frac{k\, q^{d_1 + d_2}}{\zeta_q(2)^2}, \quad k = \prod_P \left(1 + \frac{1}{(|P|+1)^2} \right)$$

$d_1, d_2 \to \infty$

$$\frac{\# \{ F \in \mathcal{F}_{(d_1, d_2)} : F(a_i) = \alpha_i \quad 1 \leq i \leq q \}}{1}$$

$$\sim \frac{k\, q^{d_1 + d_2}}{\zeta_q(2)^2} \left(\frac{2}{q+2} \right)^m \left(\frac{q}{3(q+2)} \right)^{q-m}$$

where $m = \#\, \alpha_i$ which are 0.

Thm (Bucur - D - Feigon - Lalin)

$$\frac{\# \{ C \in \mathcal{H}^{(d_1, d_2)} : \# C(\mathbb{F}_q) = m \}}{\# \mathcal{H}^{(d_1, d_2)}} = \text{Prob}\left(\sum_{i=1}^{q+1} X_i = m \right)$$

where

$$X_i = \begin{cases} 0 & \frac{2q}{3(q+2)} \\ 1 & \frac{2}{q+2} \\ 3 & \frac{q}{3(q+2)} \end{cases}$$

Now what are the stats for $\mathcal{H}_{g,3}$?

$$\# \mathcal{H}_{g,3} = \sum_{\substack{d_1 + 2d_2 \equiv 0 (3) \\ d_1 + d_2 - 2 = g}} \# \mathcal{H}^{(d_1, d_2)}$$

we only have $d_1, d_2 \to \infty$

need estimates when

$d_1 + d_2 \to \infty$

What are the prob for $\mathcal{C}_{g,3}$?

Suppose one looks at $Y^3 = F(X)$, $F(X)$ cube free of degree d

(Wood $Y^r = F(X)$, Xiong $Y^e = F(X)$)

$$\lim_{d\to\infty} \frac{\#\{F \in \mathcal{F}_d^{\text{cubefree}} : \# C(\mathbb{F}_q)_{\text{aff}} = m\}}{\# \mathcal{F}_d^{\text{cubefree}}}$$

$$\sim \text{Prob}\left(\sum_{i=1}^{q} X_i = m \right)$$

where $X_i = \begin{cases} 0 & \dfrac{2}{3(q^{-2}+q^{-1}+1)} \\ 1 & \dfrac{q^{-2}+q^{-1}}{q^{-2}+q^{-1}+1} = \dfrac{q^2-1}{q^3-1} \\ 3 & \dfrac{1}{3(q^{-2}+q^{-1}+1)} \end{cases}$

What is the distribution for $\mathcal{H}_{g,3}$?

Counting curves C with n points $\longleftrightarrow$

counting $\mathbb{F}_q(C)$ with given ramification/splitting conditions at the primes of degree 1 of k.

$$Z_C(u) = \exp\left(\sum_{n=0}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} u^n \right)$$

$$= Z_{\mathbb{F}_q(C)}(u) = \prod_{p \in S_k} \left(1 - u^{\deg p \, f(p)}\right)^{-r(p)}$$

$K = \mathbb{F}_q(C)$

$|$

k

and taking logs and equation coefficients

$$\#C(\mathbb{F}_{q^n}) = \sum_{\substack{f(p)\deg p \mid n \\ p \in S_k}} r(p) f(p) \deg(p)$$

For $C \in \mathcal{H}_{g,3}$, $K = \mathbb{F}_q(C)$

$$\#C(\mathbb{F}_q) = 3\,\#\{P \in S_k : \deg P = 1 \ \&\ p \text{ splits in } k\}$$
$(n=1)$
$$+\ 1\,\#\{P \in S_k : \deg P = 1 \ \&\ p \text{ ramifies in } k\}$$

Then

$$\langle \#C(\mathbb{F}_q) \rangle_{\mathcal{H}_{3,g}} = \frac{1}{\#\text{field}} \sum_{\substack{K \text{ of genus } g \\ \text{cyclic of} \\ \text{order } 3/\mathbb{F}_q(X)}} 3 \sum_{P} + 1 \sum_{P}$$

$$= \sum_{P \text{ of degree } 1} \left[\frac{\#E_3(k, g, P, \text{Ram})}{\#E_3(k, g)} + \frac{3\,\#E_3(k, g, P, \text{split}}{\#E_3(k, g} \right]$$