L-fcts over function Fields

Number fields	Function Fields
$\mathbb{Q}$	$k = \mathbb{F}_q(T)$
$\mathbb{Z}$	$A = \mathbb{F}_q[T]$
p prime	$P(x)$ irreducible polynomial (monic)
$\lvert n \rvert$	$q^{\deg F}$, $F(x) \in \mathbb{F}_q[T]$

"Riemann zeta fct" / $\mathbb{F}_q(T)$

$$\zeta_q(s) = \sum_F \frac{1}{|F|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1}$$

$$= \sum_{d=0}^{\infty} \frac{q^d}{q^{ds}} = \sum_{d=0}^{\infty} (q^{1-s})^d = \frac{1}{1-q^{1-s}}$$

David Z

PNT over $\mathbb{F}_q[T]$

a_d = # prime polynomials of degree d

$$\zeta_q(s) = \prod_{d=1}^{\infty} \left(1 - \frac{1}{q^{ds}}\right)^{-a_d} = \frac{1}{1-q^{1-s}}$$

$u = q^{-s}$

$$\prod_{d=1}^{\infty} \left(1 - \frac{1}{u^{-d}}\right)^{-a_d} = \frac{1}{1-qu}$$

$$u \sum_{d=1}^{\infty} a_d \frac{d}{du} \log(1 - u^d) = u \frac{d}{du} \log(1-qu)$$

$$\iff \sum_{d=1}^{\infty} \frac{a_d \, d \, u^d}{1-u^d} = \frac{qu}{1-qu}$$

writing the geometric series

$$\sum_{d=1}^{\infty} d a_d \sum_{n=1}^{\infty} u^{dn} = \sum_{d=1}^{\infty} (qu)^d$$

<u>Equating coeffs</u> $\quad q^d = \sum_{n \mid d} n a_n$

$$\iff a_n = \frac{1}{n} \sum_{d \mid n} \mu(d)\, q^{n/d}$$

<u>Thm</u> $\quad a_n = \dfrac{q^n}{n} + O\left(\dfrac{q^{n/2}}{n} + \dfrac{q^{n/3}}{n} \sum_{d \mid n} \cdots \right)$

<u>lemma</u> $\quad \#\mathcal{F}_d = \begin{cases} q^d - q^{d-1} = \dfrac{q^d}{\zeta_q(2)} & d \geq 2 \\ q^d & d = 0, 1 \end{cases}$

<u>proof</u> $$\zeta_q(s) = \zeta_q(2s) \sum_{n=1}^{\infty} |\#\mathcal{F}_d|\, q^{-ds}$$

...

General Function fields $K/k = \mathbb{F}_q(T)$

<u>Ex</u> $k(\sqrt{D(x)})$, $k(\sqrt[\ell]{D(x)})$

$\mathcal{P}_K$ = set of primes of K

= set of DVR R containing $\mathbb{F}_q$

and <u>st</u> $\mathrm{SF}(R) = K$

P = maximal ideal of R is the prime

$$|P| = \left| {}^{R/P}\big/_{\mathbb{F}_q} \right|$$

$k \quad \mathbb{F}_q(x) \quad \mathcal{P}_k = \{P \text{ irreducible polynomial}\}$
$\cup \{\infty\}.$

coming from $R' = A[\frac{1}{x}] \quad P = (\frac{1}{x})$

<u>Divisors</u> $\mathcal{D}_k$: Free abelian group generated by the primes

$$D = \sum a_P (P) \qquad \deg D = \sum a_P \deg P$$

$$|D| = q^{\deg D}$$

$$|D_1 + D_2| = |D_1| \, |D_2|$$

$\mathcal{D}_k^+$ = effective divisors $a_P \geq 0$.

Thm Let χ be non trivial Dirichlet character to the modulus M.

Then $L(s,\chi) = \sum_{F} \frac{\chi(F)}{|F|^s}$ is a polynomial

in q^{-s} of degree at most $\deg M - 1$.

proof $$L(s,\chi) = \sum_{n=0}^{\infty} A(n,\chi)\, q^{-ns}$$

$$A(n,\chi) = \sum_{\deg F = n} \chi(n)$$

Suppose $n \geq \deg M$, then each residue class mod M is represented **exactly** $q^{n-\deg M}$ times

$$A(n,\chi) = q^{n-\deg M} \underbrace{\sum_{\substack{r \\ \bmod M}} \chi(r)} = 0$$

This is true in general.

__Thm__ Let $\zeta_K(s) = \dfrac{P_K(q^s)}{(1-q^{-s})(1-q^{1-s})}$

Then P_K is a polynomial of degree at most $2g$ in q^{-s}.

__proof__ From Riemann Roch thm,

let $b_n(K)$ = # of effective divisors of degree n

Then by Riemann Roch thm,
for $n > 2g-2$,

$$b_n(k) = h_k \frac{q^{n-g+1}-1}{q-1}$$

and $Z_K(u) = \sum_{n=1}^{\infty} b_n u^n$ and use

<u>Then</u> $Z_K(u) = \dfrac{P_K(u)}{(1-u)(1-qu)}$

Look at the zeroes of $Z_K(u)$.

$$P_K(u) \quad \prod_{i=1}^{2g} (1 - u\,\alpha_j(k))$$

$\zeta_K(s) = 0 \iff q^{-s} = \alpha_j(K)^{-1}$ for some j

<u>RH</u> $s = 1/2 \iff |\alpha_j(K)| = \sqrt{q}$

<u>Thm</u> (Weil, Stephanov-Bombieri)

The zeroes of $\zeta_K(s)$ have $Re(s) = 1/2$

$\iff |\alpha_j(K)| = \sqrt{q}$

<u>Thm</u> $a_n(K) = \#\{P \in S_K \text{ of degree } n\}$

$$= \frac{q^N}{N} + O(q^{N/2})$$

Another formulation for $Z_K(u)$

$$-\log(1-u) = \sum_{n=1}^{\infty} \frac{u^n}{n}$$

① $$\log Z_K(u) = \log \prod_{d=1}^{\infty} (1-u^d)^{-a_d(K)}$$

$$= \sum_{d, m} a_d(K) \frac{u^{dm}}{m}$$

$$= \sum_{n=1}^{\infty} N_n(K) \frac{u^n}{n}$$

$$N_n(K) = \sum_{d \mid n} d\, a_d(K)$$

② $\log Z_K(u) = \log\left(\frac{P_K(u)}{(1-u)(1-qu)}\right)$

$$= \sum \frac{u^n}{n} + \sum \frac{(qu)^n}{n} - \sum_{j=1}^{2g} \sum_{n} \frac{\alpha_j(C)^n}{n} u^n$$

<u>Then</u> $N_n(K) = \sum_{d \mid n} d\, a_d(K)$

$= \#$ of primes of degree 1 in $\underline{\mathbb{F}_{q^n} K}$

<u>① + ②</u> $N_n(K) = q^n + 1 - \sum_{j=1}^{2g} \alpha_j(C)^n$

Function Field	Curves over $\mathbb{F}_q$
$\mathbb{F}_q(C) = \mathbb{F}[x,y]/(F(x,y))$	C smooth projective curve over $\mathbb{F}_q$ $F(x,y)=0$
$\mathbb{R}(\sqrt{D(x)})$	$y^2 = D(x)$
primes in $\cancel{\mathbb{F}_q(C)}$ $K = \mathbb{F}_q(C)$ $\longleftrightarrow$	Galois orbits of points on $C(\overline{\mathbb{F}_q})$
ex $K = \mathbb{F}_q(x)$ $P(x)$ with roots $\theta_1, \ldots, \theta_d$ $\longleftrightarrow$	$\{\theta_1, \ldots, \theta_d\}$ Galois orbit in $\mathbb{P}^1(\overline{\mathbb{F}}_q)$

primes of degree 1 — points in $C(\mathbb{F}_q)$

prime of degree 1 in $\mathbb{F}_{q^n}K$ — points in $C(\mathbb{F}_{q^n})$

Then $\underline{Z_K(u)} = \cancel{\sum N_n(K)\, u^n}$

$$= \exp\Bigg(\sum_{n=1}^{\infty} \underbrace{\#C(\mathbb{F}_{q^n})}_{=\,N_n(K)} \frac{u^n}{n} \Bigg)$$

$$\overset{\text{def}}{=} \underline{Z_C(u)}$$