<u>Katz Sarnak</u> statistics for zeros of L-fcts in families should match the corresponding statistics for eigen values of large random matrices

Montgomery (1974).

$\zeta(\tfrac{1}{2}+it) \quad t\in\mathbb{R}$, RH

$$N(T) = \#\{ 0 < t < T : \zeta(\tfrac{1}{2}+it) = 0\}$$
$$\sim \frac{T\log T}{2\pi}$$

$\zeta(\tfrac{1}{2}+i\gamma) = 0 \qquad \tilde{\gamma} = \dfrac{\gamma\log\gamma}{2\pi}$

<u>Pair correlation</u> $\qquad \dfrac{1}{N(T)} \sum\limits_{0<\gamma,\gamma'<T} f(\tilde{\gamma}' - \tilde{\gamma})$

Dav 1

Thm $\displaystyle \frac{1}{N(T)} \sum_{0<\gamma,\gamma'<T} f(\tilde{\gamma}-\tilde{\gamma}') \xrightarrow{T\to\infty} \int_{\mathbb{R}}$

$$\longrightarrow \int_{\mathbb{R}} f(x)\left(1-\frac{\sin^2(\pi x)}{(\pi x)^2}\right)dx$$

for some test function f st $\mathrm{supp}(\hat{f}) \subseteq (-\delta,\delta)$

$U(N) = N\times N$ unitary matrices $\quad U^*U = UU^* = I_N$

$\lambda_j(U) = e^{i\theta_j(U)} \quad j=1,\dots,N$

$$C_f(U) = \sum_{\substack{1\le j,k\le N\\ j\ne k}} f\left(\frac{N}{2\pi}\theta_j - \frac{N}{2\pi}\theta_k\right)$$

scaling limit

Thm $\displaystyle \int_{U(N)} C_f(U)\,dU \xrightarrow{N\to\infty} \int_{-\infty}^{\infty} f(x)\left[1-\frac{\sin^2(\pi x)}{(\pi x)^2}\right]dx$

Evidence for Montgomery's conjecture

- Numerical Evidence (Odlyzko)
- Generalisation to zeroes of $\pi \in GL(m)$ (Rudnick - Sarnak)
- This was proven over function fields when $q \to \infty$

<u>Another statistics</u> One level density

(study of the low lying zeroes)

① Family of L-fcts of elliptic curves / $\mathbb{Q}$. $L(s,E)$

② Family of L-fcts attached to quadratic Dirichlet character $L(s,\chi_d)$

$$W_f(E) = \sum_{L(1+i\gamma_E, E)=0} f\left(\frac{\gamma_E \log N_E}{2\pi}\right) \qquad f \in \mathcal{S}(\mathbb{R})$$

$L(s, E)$

$$\Lambda(s, E) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L(s, E) = \Lambda(2-s, E)$$

$L(1+i\gamma, E)$

One level density

$$\langle W_f(E) \rangle_{\mathcal{F}(X)} = \frac{1}{\# \mathcal{F}(X)} \sum_{E \in \mathcal{F}(X)} W_f(E)$$

where $\mathcal{F}(X)$ is a family of elliptic curves
<u>st</u> $\operatorname{cond}(E) \sim X$

<u>Conj</u> (Katz Sarnak)

$$\langle W_f(E) \rangle_{\mathcal{F}(X)} \xrightarrow{X \to \infty} \int_{\mathbb{R}} f(x)\, W_G(x)\, dx$$

where $W_G(x)$ depends on the symmetry type of the family.

$$W_G(x) = \begin{cases} 1 & U \quad \text{unitary} \\ 1 - \frac{\sin(2\pi x)}{2\pi x} & USp \quad \text{symplectic} \\ 1 + \frac{1}{2}\delta_0(x) & O \quad \text{orthogonal} \\ 1 + \delta_0(x) \quad \frac{\sin(2\pi x)}{2\pi x} & SO(\text{odd}) \\ 1 + \frac{\sin(2\pi x)}{2\pi x} & SO(\text{even}) \end{cases}$$

Those an thee scaling density for one level density on the matrix groups. $G = O(N)$

$$\lim \int_{O(N)} W_f(u)\, du \xrightarrow[N \to \infty]{} \int_{\mathbb{R}} f(x) \left[1 + \frac{1}{2}\delta_0(x) \right] dx$$

$$= \hat{f}(0) + \frac{1}{2} f(0)$$

$$\Lambda(s,E) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s)\, L(s)$$

$$\frac{L'}{L}(s) = \sum_{p,k} \frac{\alpha_p^k + \bar{\alpha}_p^k}{k p^{ks}}\, p^s$$

$$\alpha_p^2 + \bar{\alpha}_p^2 = a_p^2 - 2p$$

<u>Remark</u> The Fourier transforms of O, SO (odd) & SO (even) agree for $|u| < 1$

<u>First family</u> $y^2 = x^3 + ax + b$ $\quad |a| \leq X^{1/3}, \ |b| \leq X^{1/2}$

<u>st</u> $\Delta_E \asymp X$

<u>Explicit formulas</u> (Weil)

$$\sum_{\gamma_E} f\left(\frac{\gamma_E \log N_E}{2\pi}\right) = \hat{f}(0) + \frac{1}{2} f(0)$$

$$- \sum_p \frac{2 \log p}{p \log N_E} \hat{f}\left(\frac{\log p}{\log N_E}\right) a_p(E) + O\left(\frac{\log \log N_E}{\log N_E}\right)$$

<u>coming from</u> $\displaystyle\int_{(2+\varepsilon)} \frac{\Lambda'}{\Lambda}(s)\, h(s)\, ds$

Then $\langle W_f(E) \rangle_{\mathcal{F}(X)} = \hat{f}(0) + \frac{1}{2} f(0)$

$$+ \frac{2}{\log X} \sum_{p \le X^{\delta}} \frac{\log p}{p} \sum_{E \in \mathcal{F}(X)} a_p$$

provided $\hat{f}$ has support in $(-\delta, \delta)$

Thm (Young)

$$\langle W_f(E) \rangle_{\mathcal{F}(X)} = \hat{f}(0) + \frac{1}{2} f(0)$$

for support $(\hat{f}) \subseteq \left(-\frac{7}{9}, \frac{7}{9}\right)$

This confirms O symmetries (well, but also SO(odd) or SO(even) would work)

~~Thm~~ **Cor** Average analytic rank $\leq \frac{1}{2} + \frac{9}{7} = \frac{25}{14} < 2$

Let's look at an example where the symmetry type is not clear a priori.

$$E_t : y^2 = x^3 + tx^2 - (t+3)x + 1, \quad t \in \mathbb{Z}$$

Washington-Rizzo.

Rank 1 over $\mathbb{Q}(t)$

$W(E_t) = -1 \quad \forall t \in \mathbb{Z}$.

<u>Thm</u> (Miller) $\mathcal{F}(X) = \{E_t : t \sim X^{1/4}\}$

forced zero

$$\langle W_f(E_t) \rangle_{\mathcal{F}(X)} \sim \hat{f}(0) + \frac{3}{2} f(0) = f(0) + \hat{f}(0) + \frac{1}{2} f(0)$$

for $\operatorname{supp}(\hat{f}) \subseteq \left(-\frac{1}{3}, \frac{1}{3}\right)$.

This agrees with $W(x) = \delta_0(x) + SO(\text{odd})$

Thm (Huyhn - Parks - D).

Assume the ratio conjectures. Then

$$\langle W_f(E_t) \rangle_{\mathcal{F}(X)} \sim \int_{-\infty}^{\infty} f(x)\left[\delta_0(x) + 1 + \frac{\sin(2\pi x)}{2\pi x}\right] dx$$

ie $W(x) = \cancel{SO^+}\ \delta_0(x) + SO(\text{even})(x)$

Another family $L(s, \chi_d)$, $\chi_d = \left(\frac{d}{n}\right)$, $d \sim X$

Thm (Sarnak, Ozlek Snyder)

$$\langle W_f(d) \rangle_{D(X)} = \int_{\mathbb{R}} f(x)\left[1 - \frac{\sin(2\pi x)}{2\pi x}\right] dx$$

when ~~spt~~ $\mathrm{supp}(f) \subseteq (-2, 2)$.

symplectic symmetries

Try to compute the n-level density for the same family

<u>Thm</u> (Rubinstein)

$$\langle W_f^{(n)}(d) \rangle_{D(x)} \longrightarrow \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \, W^{(n)}(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

when $\hat{f}(u_1, \dots, u_n)$ has support contained in $\sum |u_i| < 1$

Try to extend to $\sum |u_i| < 2$.

<u>Gao</u>, $n=3,4$, <u>Miller</u> $n=5,6$

↳ $$\langle W_f^{(n)}(d) \rangle_{D(x)} = A(f) + o(1)$$

Thm This was proven for all n by

Entin- Roddity- Gershon - Rudnick

using hyperelliptic curves / $\mathbb{F}_q$

ie Katz - Sarnak equidistribution thm !