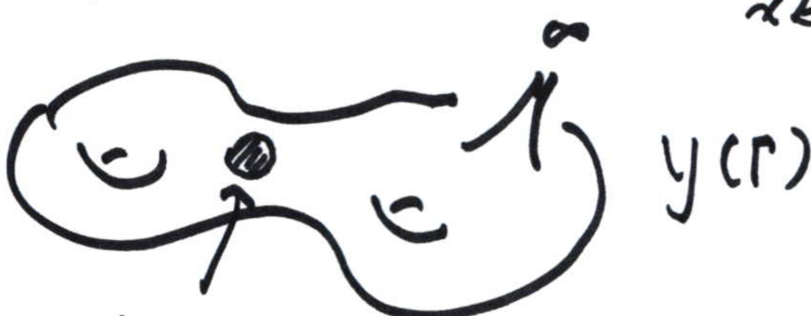
p-divisible groups

$\mathcal{Y}(\Gamma)/W = W(\bar{\mathbb{F}}_p)$

$\hat{\mathcal{O}}_{\mathcal{Y}(\Gamma),x}$, $x \in \mathcal{Y}(\Gamma)(\bar{\mathbb{F}}_p)$

$x \longleftrightarrow (E_0, P, Q)$



$\hat{\mathcal{O}}_{\mathcal{Y}(\Gamma),x}$ = functions on this region

region where $E \bmod p = E_0$

To deform E_0 is to deform $E_0[p^\infty]$.

S scheme

G/S group scheme is...

$\mu : G \times_S G \to G$

$e : S \to G$

$i : G \to G$

Functor of points:

for $T \to S$, get $\underset{\uparrow \text{ group.}}{G(T)} = \mathrm{Hom}_S(T, G)$

$G : S\text{-Sch} \to \text{Groups.}$

<u>Ex.</u> $\circ$ Gp schemes $/\mathbb{Z}$:

$\mathbb{G}_a = \operatorname{Spec} \mathbb{Z}[T]$, $\mathbb{G}_a(R) = R$, under $+$

$\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$, $\mathbb{G}_m(R) = R^\times$, under $\times$.

$\underline{G} = \coprod_G \operatorname{Spec} \mathbb{Z}$ $\quad \underline{G}(R) = G$
(if Spec R connected.)

$\mu_N = \operatorname{Spec} \dfrac{\mathbb{Z}[T]}{T^N - 1}$ $\quad \mu_N(R) = \{x \in R \mid x^N = 1\}$

If R is a $\mathbb{Z}[\frac{1}{N}]$-alg., then

$\mu_{N,R}$ is étale

(becomes constant after passing to étale ext'n of R)

If $N = p$. $\mu_p / \mathbb{F}_p$ not étale

$\mu_{p,\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x] / \underset{(x-1)^p}{x^p - 1}$ connected

$\mu_p / \mathbb{F}_p$ is connected.

$\underset{\text{finite flat gp. scheme } / R}{E[N]}$, E/R ell. curve

étale if $\frac{1}{N} \in R$

$E[p] / \mathbb{F}_p$ is never étale.

Cartier duality.

G/S finite flat gp. sch.

$$\hat{G}(T) = \mathrm{Hom}_T(G_T, \mathbb{G}_{m,T})$$

Ex. $(\underline{\mathbb{Z}/N\mathbb{Z}})^\wedge = \mu_N$

$$E[N]^\wedge = E[N]$$

$$e_N : E[N] \times E[N] \to \mu_N.$$

Interested in system

$$E[p] \subset E[p^2] \subset E[p^3] \subset \ldots$$

this is a p-divisible gp.

<u>Defn</u>. Let p = prime, $h \geq 1$,
R a ring

A p-div. gp. $/R$ of height h is
a directed system

$$G = \lim_{n\to\infty} G_n = (G_1 \to G_2 \to G_3 \to \dots)$$

G_n = p-torsion commutative
finite flat group scheme $/R$
locally free rk p^{nh}

$$G_n \to G_{n+1} \rightsquigarrow G_n \xrightarrow{\sim} G_{n+1}[p^n] \subset G_{n+1}$$

<u>Examples</u>

$$\mathbb{Q}_p/\mathbb{Z}_p = \lim_{n\to\infty} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$$ ht 1, dim 0

$$\mu_{p^\infty} = \lim_{n\to\infty} \mu_{p^n}$$ ht 1, dim 1

$$A[p^\infty] = \lim_{n\to\infty} A[p^n]$$ ht $2g$, dim g

dim $A = g$

Let k = perfect field char p.

$W(k)$ = Witt vectors $\quad W(k)/p = k$

A Dieudonné module is a free $W(k)$-module M of rank $h < \infty$, with

$F, V : M \to M$ st....

$\sigma : k \to k$ Frob. auto. $x \mapsto x^p$ $\rightsquigarrow \sigma : W(k) \to W(k)$

F is σ-linear, $\quad V$ is σ^{-1}-linear,

$$FV = p.$$

$\exists$ anti-equivalence of categories

$$\underset{\text{p-div gps}/k}{G} \to \underset{\text{D. modules.}}{M(G)}$$

$$\text{ht } G = \text{rk } M(G)$$

$$\dim G = \dim_k M(G)/FM(G)$$

G connected $\Leftrightarrow$ $F^n M(G) \subset pM(G)$ for $n \gg 0$.

G étale $\Leftrightarrow$ F invertible.

<u>Ex</u> $k=\mathbb{F}_p$, $W(k)=\mathbb{Z}_p$

$$M(\mathbb{Q}_p/\mathbb{Z}_p)=\mathbb{Z}_p \qquad F=1,\ V=p$$

$$M(\mu_{p^\infty})=\mathbb{Z}_p \qquad F=p,\ V=1$$

If E/k, $\bar{k}=k$.

$$M(E[p^\infty])\simeq W(k)^{\oplus 2}$$

ht 2
dim 1

$$F=\begin{cases}\begin{pmatrix}1 & \\ & p\end{pmatrix} & E \text{ ordinary.}\\[2ex] \begin{pmatrix} & 1\\ p & \end{pmatrix} & E \text{ s.s.}\end{cases}$$

in ord. case:

$$E[p^\infty]\simeq \mathbb{Q}_p/\mathbb{Z}_p\oplus\mu_{p^\infty}.$$

The Serre-Tate thm., or how to deform $E_0/\mathbb{F}_p \to E/\mathbb{Z}_p$

Idea: to give $E/\mathbb{Z}_p$, only necessary to give $E \otimes \mathbb{F}_p$ and $E[p^\infty]$ (the p-div gp $/\mathbb{Z}_p$)

R = Artinian local ring, M, $R/M = \overline{\mathbb{F}}_p$

$\mathcal{E}ll_R = \{$e.c.'s over $R\}$.

$\mathcal{A}_R = \{(E_0, G, \iota)\}$, where

- $E_0/\overline{\mathbb{F}}_p$ e.c.
- G = p-div gp $/R$
- $\iota: E_0[p^\infty] \xrightarrow{\sim} G \underset{R}{\otimes} \overline{\mathbb{F}}_p$

Thm. $\mathcal{E}ll_R \to \mathcal{A}_R$

$$E \mapsto (E \underset{R}{\otimes} \overline{\mathbb{F}}_p, E[p^\infty], id.)$$

is an equivalence.

Prop. Let $W = W(\overline{\mathbb{F}}_p)$ $\quad p \nmid N$

$\mathcal{C}$ = cat. of complete local noeth W-algs, of residue field $\overline{\mathbb{F}}_p$

$x = (E_0, \underline{P}) \in Y_1(N)(\overline{\mathbb{F}}_p)$

Let $\mathcal{F} : \mathcal{C} \to \text{Sets}$

$R \mapsto \{ (G, \iota) \mid G/R$ p-div gp, $\iota : \underline{E_0[p^\infty]} \xrightarrow{\sim} G \otimes_R \overline{\mathbb{F}}_p \}$

Then $\mathcal{F}$ is representable by $\widehat{\mathcal{O}}_{Y_1(N), x}$ $\simeq W[\![t]\!]$.

Next time: E_0 s.s., get $E_0[p^\infty]$ connected ("formal")