

Lecture 2: Basic Facts

Hyperbolic Laplacian $k \in \mathbb{R}$

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

$$z := x+iy \in \mathbb{H}$$

Def. A real analytic fn $M: \mathbb{H} \rightarrow \mathbb{C}$ is a wjt/k harmonic Maass form on Γ if:

$$1) \quad M\left(\frac{az+b}{cz+d}\right) = " (cz+d)^k M(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$2) \quad \partial_k M = 0$$

3) There is a polynomial $P_M(q^{-1}) \in \mathbb{C}[q^{-1}]$ s.t.

$$M(z) - P_M(z) = O(e^{-\delta y})$$

for some $\delta > 0$ as $y \rightarrow \infty$. [Principal part at ∞].

Ex. $j(z) - 744 = q^{-1} + 196884 \frac{1}{q} + \dots$

$$P_j(j-744) = q^{-1}$$

(Principal Parts
at ∞)

Note: E_2^∞ is not a HMF in this sense.

Fourier Expansion ($k \in \frac{1}{2} \mathbb{Z}$ (Simplicity))

Incomplete Γ-function:

$$\Gamma(\alpha; x) := \int_x^\infty e^{-t} t^{\alpha-1} dt.$$

Lemma. Suppose $f \in H_{2-k}(\mathbb{C})$, where $(c_i) \in \mathbb{C}$.

At ∞ f has an expansion of the form

$$f = \sum_{n=-\infty}^{\infty} c_f^+(n) q^n + \sum_{n<0} \bar{c}_f(n) \left[(k-1) 4\pi I_{\text{analy}} \right]^n.$$

↓ ↓ ↑

Fourier coefficients I_{analy}

\mathbb{C}

Proof. Ex #1. Problem Sheet.

Terminology $f \in H_{2-k}(\Gamma)$

$$f^+ = \sum_{n>-\infty} c_f^+ n! q^n \quad \text{"Holomorphic part" of } f.$$

$$f^- = \sum_{n<0} c_f^-(n) \Gamma(k-n, 4\pi i ny) q^n \quad \text{"Nonholomorphic Part of } f\text{"}$$

Question: What is the significance of $c_f^+(n) + c_f^-(n)$?

First Question: How is $H_{2-k}(\Gamma)$ related to classical modular forms!

EZ Answer:

$$H_{z-k}(\Gamma) \supseteq M_{z-k}^{\dagger}(\Gamma)$$

wedkly hol. m.f.s

(pole) allowed but must
be supported at
cospi]

EZ Answer #2

If $f, f_1, f_2 \in H_{z-k}(\Gamma)$ s.t.

$$\bar{f_1} = \bar{f_2}$$

$$\Rightarrow f_1 - f_2 = f_1^+ - \bar{f_2} \in M_{z-k}^{\dagger}$$

Deeper Answer (x_i -operator)

$$l_w := 2i \cdot y^w \cdot \frac{\partial}{\partial \bar{z}}$$

Exercise: $l_w(f) = l_w(f^\#)$ ✓

Lemma Suppose $f \in H_{2-k}(\mathbb{C})$, then

$$l_{2-k}(f) \in S_k(\mathbb{C}).$$

Moreover we have that

$$l_{2-k}(f) = l_{2-k}(f^\#) = - (4\pi)^{k-1} \sum_{n \geq 1} c_f^{-n} n^{k-1} q^n.$$

and

$$l_{2-k} : H_{2-k} \longrightarrow S_k.$$

Natural Question: Given $F \in S_k(\mathbb{D})$,

how do we:

- 1) Find $f \in H_{2k}(\mathbb{C})$ s.t.

$$\mathcal{L}_{2-k}(f) = F.$$

[only may answers ...]

- 2) How do you find a "good f "

~~$f = f^+ + f^-$~~ ↴

$f = f^- + f^+$

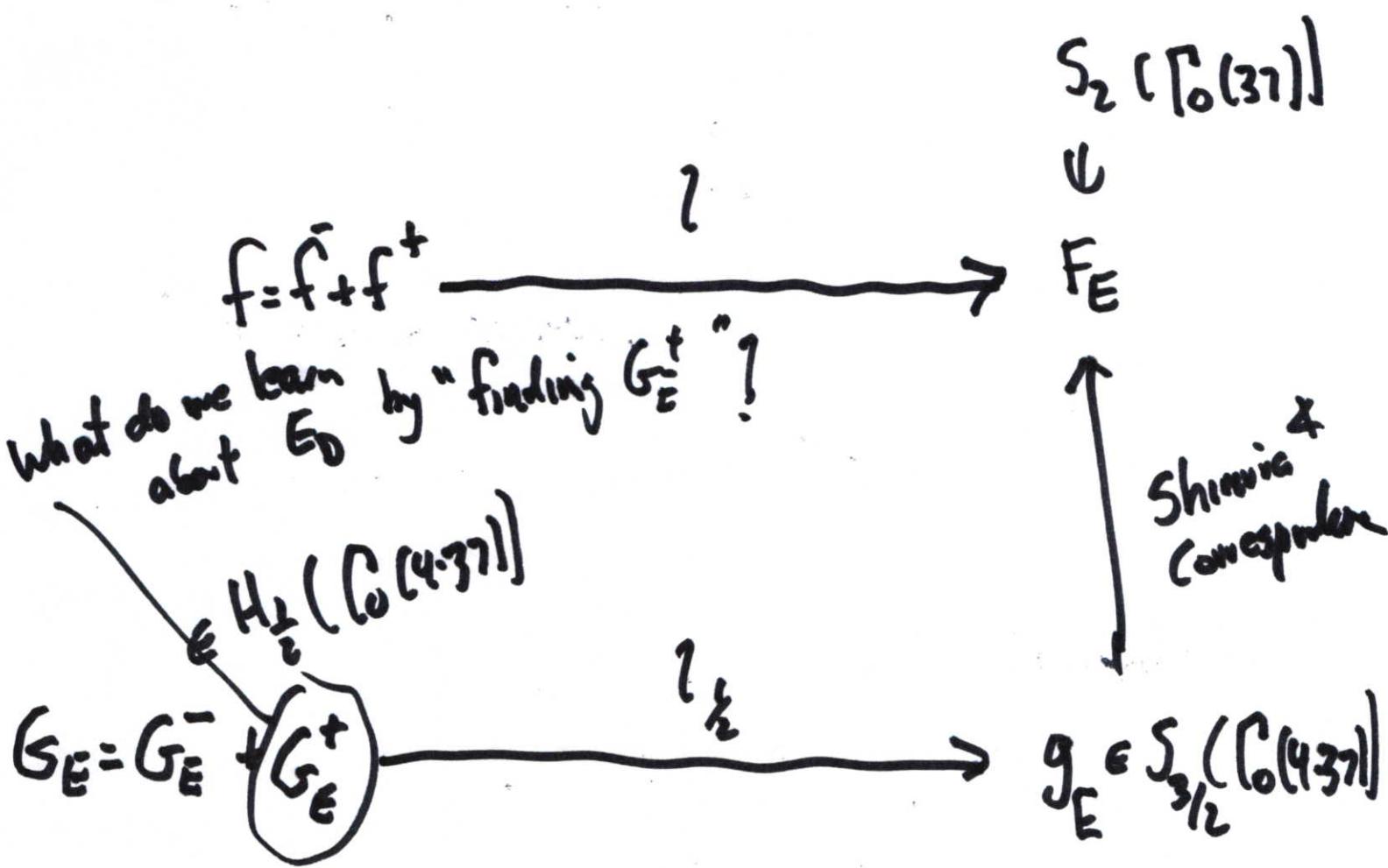
holomorphic fun.

Explicit Example

E(4) conductor 37

$$F_{E(z)} \in S_2(\Gamma_0(37))$$

"Modularity"



* (Waldspurger...) Coefficients of g_E

interpolate: $L(E_D, i)$ as D varies over f.d.

\nearrow
BSD...

Periods For Modular Forms

Fact: Suppose $f \in H_{2-k}(\Gamma)$ and $\mathcal{I}_{2-k}(f) = \bar{F} \in S_k(\Gamma)$

Then we have $\mathcal{I}^- = \text{"period integral of } F\text{"}$

↑
"Eichler integral"

+
"Period Polynomial"

Lemma.

$$i(2\pi n)^{1-k} \int_{-\frac{i}{2}}^{i\infty} \frac{e^{2\pi ny\tau}}{(-i(\tau+z))^{2-k}} d\tau$$

Hecke Eigenforms Suppose $F \in S_k(\Gamma_0(1))$ is a Hecke eigenform. Then the periods of F are,

for $0 \leq n \leq k-2$, the numbers

$$L(F, n+1) := \frac{(2\pi)^{n+1}}{n!} \int_0^\infty f(it) t^n dt.$$

EZ Thm. Suppose that $1 \leq k \in \mathbb{Z}$, and suppose

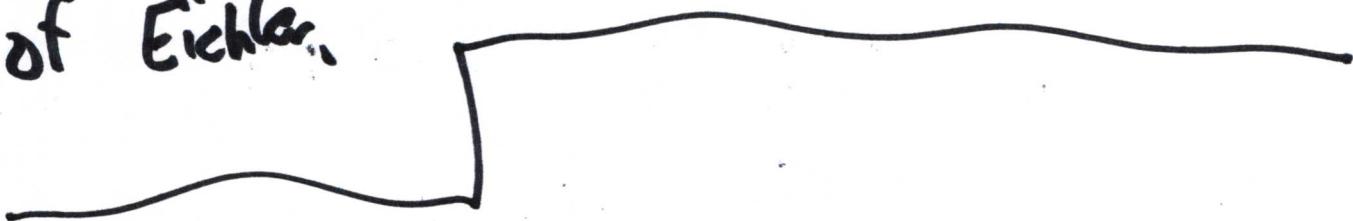
$f \in H_{2-k}(\Gamma_0(1))$ s.t. $T_{2-k}(f) = F \in S_{2k}$, a normalized Hecke eigenform. Define $P_{F,f}(z)$ by

$$P_{F,f}(z) := \frac{(4\pi)^{k-1}}{(2\pi)^{k-1}} \cdot \left[f^+(z) - f^+(-\frac{1}{z}) z^{k-2} \right].$$

Then we have $\bar{P}_{F,f}(\bar{z}) = \sum_{n=0}^{k-2} \frac{L(F, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n} \in \mathbb{C}[[z]]$.

EZ Thm: In integral wt, the "obstruction to modularity" (i.e. $P_{F,F}$) is the "period polynomial"

of Eichler.



Q: How is EZ Thm generalized when $K \in \mathbb{Z} - \mathbb{Z}$?

No such thing as a half-int. weight period polynomial...

Example. $F = \Delta \in S_{12}(\mathbb{P}_0(1))$.

We know

$$\mathcal{I}_{-10} : H_{-10} \longrightarrow S_{12}.$$

$$f = f^- + f^+ \rightarrow 0$$

"By the method of Poincaré series" f is a best choice for f .

It turns out

$$c_{f^+}^+(1)$$

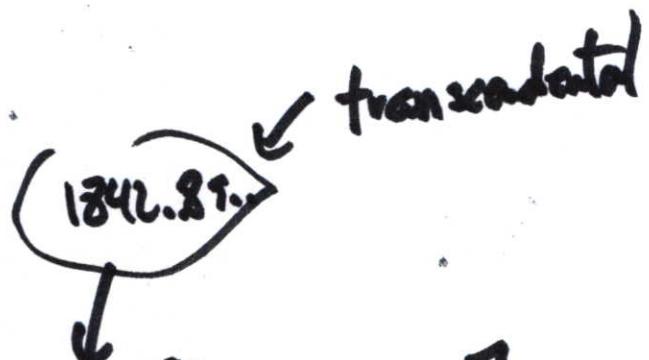
$$\text{!!! } f^+ = q^{-1} - \frac{65520}{691} - (842.899\ldots q - 23274.02\ldots q^2 - \dots)$$

and it is really very ugly!

Q: Why f^+ is a best choice?

How do we see δ from $f^+?$

A: Renormalization



$$\hat{f}^+(z) := 11! \left[f^+(z) - c_{f^+}^{(1)} \sum_{n=1}^{\infty} \epsilon^{(n)} n^{-11} \right] q^n$$

$\Rightarrow \hat{f}^+(z)$ still looks messy..

$$\Rightarrow \hat{\hat{f}}^+(z) := \left(q \frac{d}{dq} \right)^{11} \hat{f}^+(z) = \text{integer coeffs.}$$

Exercise: If p is an ordering prime for Δ , then

$$\lim_{n \rightarrow \infty} \frac{\hat{f}^+(U(p^n))}{\hat{C}^+(p^n)} = \Delta. \quad (\text{p-adically})$$

Remarks:

1) I believe $C_f^+(1)$ is transcendental.

2) "In full case"

$C_f^+(1)$ is transcendental \Rightarrow Lehmer's Conjecture.

3) If $F \in S_k(\mathbb{P}_{\text{ND}})$ s.t. F has CM, then

we know that the "best" $f \in H_{2-k}(\mathbb{P}_{\text{ND}})$

s.t.

$$L_{2-k}(f) = F$$

has $C_f^+(n)$ are algebraic integers.

Next Time:

- Traces of Singular Moduli.
- Borcherds Products
- $L'(E_D, 1)$, $L(E_D, 1)$ for E/\mathbb{Q} .

by making use of the surjectivity of $\mathcal{L}_{\frac{2-k}{2}}$