

# Project description

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## 0 Outline

The Grothendieck-Ogg-Shafarevich (GOS) formula (*cf.* [SGA5, Exposé X]) is one of the most classical results in geometric ramification theory. It describes the Euler characteristic of a smooth  $\ell$ -adic sheaf  $\mathcal{F}$  on a smooth curve over an algebraically closed field in terms of a local ramification invariant of  $\mathcal{F}$  called the Swan conductor. In [KS08], Kato and Saito obtained a higher-dimensional generalization of it. The first aim of this project is to find some concrete examples of their rather abstract formula (for a detailed plan, see Section 1).

The second and main theme of our project is to consider an arithmetic variant of the GOS formula. Let  $V$  be a complete discrete valuation ring with algebraically closed residue field. Put  $S = \text{Spec } V$  and consider a scheme  $X$  separated of finite type over  $S$  whose generic fiber  $X_\eta$  is smooth. For a smooth  $\ell$ -adic étale sheaf  $\mathcal{F}$  on  $X_\eta$ , we can attach the nearby cycle complex  $R\psi\mathcal{F}$ , which is an object of the derived category  $D_c^b(X_s, \mathbb{Q}_\ell)$  of  $\ell$ -adic sheaves over the special fiber of  $X$  (*cf.* [SGA7, Exposé XIII]). In our project, we will study the compactly supported cohomology  $H_c^i(X_s, R\psi\mathcal{F})$ , especially its Euler characteristic  $\chi_c(X_s, R\psi\mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H_c^i(X_s, R\psi\mathcal{F})$ . If  $X$  is proper over  $S$ , the proper base change theorem tells us that  $H_c^i(X_s, R\psi\mathcal{F})$  is isomorphic to  $H^i(X_\eta, \mathcal{F}_\eta)$ , the  $\ell$ -adic cohomology of the generic fiber. In this case, no contribution of ramification of  $\mathcal{F}$  appears. However, if  $X$  is not proper over  $S$ , the nearby cycle cohomology  $H_c^i(X_s, R\psi\mathcal{F})$  should reflect arithmetic ramification of  $\mathcal{F}$ . Note that  $H_c^i(X_s, R\psi\mathcal{F})$  is not necessarily isomorphic to  $H_c^i(X_\eta, \mathcal{F}_\eta)$ . The Euler characteristic  $\chi_c(X_\eta, \mathcal{F}_\eta)$  of the latter can be described by the classical GOS formula, and only reflects ramification of  $\mathcal{F}$  on the generic fiber.

In the language of rigid geometry, the nearby cycle cohomology  $H_c^i(X_s, R\psi\mathcal{F})$  can be interpreted as follows. Let  $X^\wedge$  be the formal completion of  $X$  along  $X_s$  and  $\mathfrak{X}$  the rigid generic fiber of  $X^\wedge$  in the sense of Raynaud. Then,  $H_c^i(X_s, R\psi\mathcal{F})$  is naturally isomorphic to the étale cohomology  $H_c^i(\mathfrak{X}_\eta, \mathcal{F}_\eta^{\text{rig}})$  of the rigid space  $\mathfrak{X}$  with a naturally induced coefficient. In this context, when  $\dim X_\eta = 1$ , Huber [Hub01] proved a formula of GOS type<sup>1</sup>:

$$\chi_c(X_s, R\psi\mathcal{F}) = \chi_c(\mathfrak{X}_\eta, \mathcal{F}_\eta^{\text{rig}}) = \text{rank } \mathcal{F} \cdot \chi_c(\mathfrak{X}_\eta, \mathbb{Q}_\ell) - \sum_{x \in \mathfrak{X}_\eta^c \setminus \mathfrak{X}_\eta} \text{Sw}_x \mathcal{F}.$$

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<sup>1</sup>Precisely speaking, Huber considered a locally constant torsion sheaf  $\mathcal{F}$ .

Here  $\mathfrak{X}_{\bar{\eta}}^c$  is the so-called universal compactification of  $\mathfrak{X}_{\bar{\eta}}$ . To a point  $x$  on the boundary  $\mathfrak{X}_{\bar{\eta}}^c \setminus \mathfrak{X}_{\bar{\eta}}$  of the universal compactification, a valuation field  $\kappa(x)$  of rank 2 is naturally attached, and a local ramification invariant  $\text{Sw}_x \mathcal{F}$  at  $x$ , an analogue of the Swan conductor, is defined in a similar way as in the classical case (for example, we can use the natural ramification filtration on the Galois group of  $\kappa(x)$ ). This formula provides a powerful method to study the cohomology of some arithmetic curves, such as the Lubin-Tate tower for  $\text{GL}(2)$  (*cf.* [Wew05]).

In this project, we will try to understand the formula above in the style of Kato-Saito's ramification theory, and generalize it to a higher-dimensional case. For a detail, see Section 2.

## 1 The case over a field

Throughout this section, let  $k$  be an algebraically closed field with characteristic  $p > 0$ . Fix a prime number  $\ell$  different from  $p$ .

### 1.1 Warming up: one-dimensional case

Let  $X$  be an irreducible proper smooth curve over  $k$  and  $U$  a non-empty open subset of  $X$ . To a smooth  $\overline{\mathbb{Q}}_{\ell}$ -sheaf  $\mathcal{F}$  on  $U$  and a point  $x \in X \setminus U$ , we can attach a ramification invariant  $\text{Sw}_x(\mathcal{F}) \in \mathbb{Z}$  called the Swan conductor (for the definition, see Saito's lecture notes). By using this invariant, the Euler characteristic  $\chi_c(U, \mathcal{F}) = \sum_{i=0}^2 (-1)^i \dim_{\overline{\mathbb{Q}}_{\ell}} H_c^i(U, \mathcal{F})$  can be computed as follows:

**Theorem 1.1 (The Grothendieck-Ogg-Shafarevich formula)**

$$\chi_c(U, \mathcal{F}) = \text{rank}(\mathcal{F}) \cdot \chi_c(U, \overline{\mathbb{Q}}_{\ell}) - \sum_{x \in X \setminus U} \text{Sw}_x(\mathcal{F}).$$

**Problem 1.2** Fix an integer  $m \geq 1$  which is prime to  $p$ . We have an étale covering of  $\mathbb{G}_m = \text{Spec } k[T, T^{-1}]$  defined by the equation  $S^m = T$  (the Kummer covering). It is a Galois covering with Galois group  $\mathbb{Z}/m\mathbb{Z}$ . Therefore, a non-trivial character  $\chi: \mathbb{Z}/m\mathbb{Z} \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$  gives rise to a smooth sheaf  $\mathcal{L}_{\chi}$  of rank 1 over  $\mathbb{G}_m$ .

- i) Consider the compactification  $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$  of  $\mathbb{G}_m$  (namely, consider the case where  $X = \mathbb{P}^1$  and  $U = \mathbb{G}_m$ ). For the points  $0, \infty \in \mathbb{P}^1 \setminus \mathbb{A}^1$ , compute the Swan conductors  $\text{Sw}_0(\mathcal{L}_{\chi})$ ,  $\text{Sw}_{\infty}(\mathcal{L}_{\chi})$ .
- ii) By using the GOS formula, compute the Euler characteristic  $\chi_c(\mathbb{G}_m, \mathcal{L}_{\chi})$ . Observe that  $H_c^i(\mathbb{G}_m, \mathcal{L}_{\chi}) = 0$  for  $i \neq 1$ , and determine the dimension of  $H_c^1(\mathbb{G}_m, \mathcal{L}_{\chi})$ .

**Problem 1.3** We have an étale covering of  $\mathbb{A}^1 = \text{Spec } k[T]$  defined by the equation  $S^p - S = T$  (the Artin-Schreier covering). It is a Galois covering with Galois group  $\mathbb{Z}/p\mathbb{Z}$ . Therefore, a non-trivial character  $\psi: \mathbb{Z}/p\mathbb{Z} \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$  gives rise to a smooth sheaf  $\mathcal{L}_{\psi}$  of rank 1 over  $\mathbb{A}^1$ .

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- i) Consider the compactification  $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$  of  $\mathbb{A}^1$  (namely, consider the case where  $X = \mathbb{P}^1$  and  $U = \mathbb{A}^1$ ). For the point  $\infty \in \mathbb{P}^1 \setminus \mathbb{A}^1$ , compute the Swan conductor  $\text{Sw}_\infty(\mathcal{F})$ .
- ii) By using the GOS formula, compute the Euler characteristic  $\chi_c(\mathbb{A}^1, \mathcal{L}_\psi)$ . Observe that  $H_c^i(\mathbb{A}^1, \mathcal{L}_\psi) = 0$  for  $i \neq 1$ , and determine the dimension of  $H_c^1(\mathbb{A}^1, \mathcal{L}_\psi)$ .
- iii) More generally, for  $f \in k[T]$ , we can consider the smooth sheaf of rank 1 over  $\mathbb{A}^1$  associated with the étale covering defined by  $S^p - S = f$ . If we identify  $f$  with the morphism  $f: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ , then this sheaf is nothing but the pull-back  $f^*\mathcal{L}_\psi$ . Can you compute  $\text{Sw}_\infty(f^*\mathcal{L}_\psi)$  and  $\chi_c(\mathbb{A}^1, f^*\mathcal{L}_\psi)$ ?

**Problem 1.4** (*cf.* [SGA4 $\frac{1}{2}$ , Sommes trig.]) Here we assume that  $k = \overline{\mathbb{F}}_q$ , where  $q = p^n$  is a power of  $p$ . Fix a non-trivial multiplicative (resp. additive) character  $\chi: \mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}_\ell$  (resp.  $\psi: \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell$ ). Then, the étale covering  $\mathbb{G}_{m, \mathbb{F}_q} \rightarrow \mathbb{G}_{m, \overline{\mathbb{F}}_q}$ ;  $x \mapsto x^{q-1}$  and  $\chi$  give a smooth sheaf  $\mathcal{L}_\chi$  of rank 1 over  $\mathbb{G}_{m, \mathbb{F}_q} = \text{Spec } \mathbb{F}_q[T, T^{-1}]$ . Similarly, the étale covering  $\mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathbb{A}_{\overline{\mathbb{F}}_q}^1$ ;  $x \mapsto x^q - x$  and  $\psi$  give a smooth sheaf  $\mathcal{L}_\psi$  of rank 1 over  $\mathbb{A}_{\mathbb{F}_q}^1 = \text{Spec } \mathbb{F}_q[T]$ . We denote the pull-back of  $\mathcal{L}_\chi$  and  $\mathcal{L}_\psi$  to  $\mathbb{G}_m = \text{Spec } k[T, T^{-1}]$  by the same symbols.

- i) Compute  $\chi_c(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$  and  $\dim_{\overline{\mathbb{Q}}_\ell} H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$  by the GOS formula.
- ii) Use the Grothendieck-Lefschetz trace formula (*cf.* [SGA4 $\frac{1}{2}$ , Rapport, Théorème 3.2]) to prove the following:

$$G(\chi, \psi) := \sum_{a \in \mathbb{F}_q^\times} \psi(a)\chi(a) = \sum_{i=0}^2 (-1)^i \text{Tr}(\text{Frob}_q; H_c^i(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)).$$

The left hand side is called the Gauss sum.

- iii) Observe that the natural map  $H_c^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi) \rightarrow H^1(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$  is an isomorphism. By this and the Weil conjecture, we may conclude that every eigenvalue  $\alpha$  of  $\text{Frob}_q$  on  $H_c^i(\mathbb{G}_m, \mathcal{L}_\psi \otimes \mathcal{L}_\chi)$  satisfies  $|\iota(\alpha)| = q^{1/2}$  for an arbitrary isomorphism of fields  $\iota: \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$ . Together with i), we obtain a well-known identity  $|\iota G(\chi, \psi)| = q^{1/2}$ .

## 1.2 Higher-dimensional case

In [KS08], the GOS formula has been generalized to the higher-dimensional case. We would like to give some concrete examples of their theory. Here we consider  $\overline{\mathbb{F}}_\ell$ -sheaves instead of  $\overline{\mathbb{Q}}_\ell$ -sheaves. In fact, the definition of the Swan conductor for  $\overline{\mathbb{Q}}_\ell$ -sheaves in [KS08] is reduced to that for  $\overline{\mathbb{F}}_\ell$ -sheaves. Fix a non-trivial character  $\psi: \mathbb{Z}/p\mathbb{Z} \rightarrow \overline{\mathbb{F}}_\ell^\times$ . For the definition of the Swan divisor and the refined Swan conductor, see

Put  $U = \mathbb{G}_m^2 = \text{Spec } k[S, T, S^{-1}, T^{-1}]$ . For  $a, b \in \mathbb{Z}$ , consider the morphism  $f_{a,b}: U \rightarrow \mathbb{A}^1$ ;  $(x, y) \mapsto x^a y^b$ . We want to compute  $\chi_c(U, f_{a,b}^*\mathcal{L}_\psi)$  by the GOS formula

**Problem 1.5** Consider the case where  $(a, b) = (1, -1)$ . Write  $f = f_{1,-1}$ .

- i) Let  $X$  be the blow-up of  $\mathbb{P}^2$  at the origin. Then  $f: U \rightarrow \mathbb{A}^1$  extends to  $\bar{f}: X \rightarrow \mathbb{P}^1$ , which gives a compactification of  $f$ . Compute the Swan divisor  $\text{sw}(f^*\mathcal{L}_\psi)$  of  $f^*\mathcal{L}_\psi$  and the refined Swan conductor with respect to this compactification (*cf.* [Kat89], [Kat94], [KS08, §5], [AS09]. Especially, [AS09, §9, 10] is a very useful reference). Observe that  $f^*\mathcal{L}_\psi$  is clean with respect to  $U \subset X$ .
- ii) Use the GOS formula to obtain  $\chi_c(U, f^*\mathcal{L}_\psi)$ .
- iii) Compute  $\chi_c(U, f^*\mathcal{L}_\psi)$  directly, and compare with the result of ii). (Hint: use the change of coordinates  $\mathbb{G}_m^2 \cong \mathbb{G}_m^2; (x, y) \mapsto (x, xy^{-1})$ )

**Problem 1.6** Next consider the case where  $(a, b) = (1, -2)$ . Write  $f = f_{1,-2}$ .

- i) Let  $W$  be the blow-up of  $\mathbb{A}^2 = \text{Spec } k[S, T]$  along the ideal  $(S, T^2)$ . Since  $W$  is not smooth over  $k$ , we take a toric resolution  $\widetilde{W}$  of  $W$ . In this case,  $\widetilde{W}$  is the blow-up of  $W$  along the strict transform of the ideal  $(S, T)$ . Then  $f: U \rightarrow \mathbb{A}^1$  extends to  $\bar{f}: \widetilde{W} \rightarrow \mathbb{P}^1$ , which gives a partial compactification of  $f$ . Compute the Swan divisor  $\text{sw}(f^*\mathcal{L}_\psi)$  of  $f^*\mathcal{L}_\psi$  and the refined Swan conductor with respect to this partial compactification. Is  $f^*\mathcal{L}_\psi$  clean with respect to  $\widetilde{W}$ ?
- ii) Find a nice compactification  $U \hookrightarrow X$  so that  $f^*\mathcal{L}_\psi$  is clean with respect to it. Compute the Swan divisor.
- iii) Use the GOS formula to obtain  $\chi_c(U, f^*\mathcal{L}_\psi)$ . To compute the intersection number, it will be convenient to use the intersection theory on toric varieties (*cf.* [Ful93, Chapter 5]).

**Problem 1.7** Extend the arguments in Problem 1.5 and Problem 1.6 to  $f_{a,b}^*\mathcal{L}_\psi$  for general  $(a, b)$ . Is it possible to apply a similar method to the case of dimension greater than 2?

**Problem 1.8** A similar but slightly different case is considered by Denef and Loeser ([DL91]). Can we recover their result [DL91, Theorem 1.2] by using the GOS formula (at least in the 2-dimensional case)?

## 2 Arithmetic case

In this section, let  $K$  be a complete discrete valuation field of mixed characteristic  $(0, p)$ . Denote by  $\mathcal{O}_K$  the ring of integers of  $K$ . We write  $v_K$  for the normalized valuation  $K^\times \rightarrow \mathbb{Z}$ . Assume that the residue field  $k$  of  $\mathcal{O}_K$  is algebraically closed.

Let us recall Huber's GOS formula for rigid curves ([Hub01]). Here we use terminology in rigid geometry. Let  $\mathfrak{U}$  be a quasi-compact rigid curve which is separated and smooth over  $K$ , and  $\mathcal{F}$  a smooth  $\overline{\mathbb{F}}_\ell$ -sheaf on  $\mathfrak{U}$ . We have the universal compactification  $\mathfrak{U}^c$  of  $\mathfrak{U}$  (*cf.* [Hub01, 5.9, Example 5.10]). For each  $x \in \mathfrak{U}^c \setminus \mathfrak{U}$ , a

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2-dimensional henselian valuation field  $\kappa(x)$  with valuation  $v_x: \kappa(x)^\times \longrightarrow \Gamma_x$  is naturally attached. The field  $\kappa(x)$  is an extension field of  $K$  and we have the following commutative diagram:

$$\begin{array}{ccc} K^\times & \xrightarrow{v_K} & \mathbb{Z} \\ \downarrow & & \downarrow m \mapsto (m,0) \\ \kappa(x)^\times & \xrightarrow{v_x} \Gamma_x \xleftarrow[\cong]{(*)} & \mathbb{Z} \times \mathbb{Z}, \end{array}$$

where  $\mathbb{Z} \times \mathbb{Z}$  is considered as a totally ordered commutative group by the lexicographic order, and  $(*)$  is an ordered isomorphism (in fact,  $(*)$  is uniquely determined by the commutative diagram above).

By using ramification theory for  $G_{\kappa(x)} = \text{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$ , we can define the Swan conductor  $\text{Sw}_x(\mathcal{F}) \in \mathbb{Z}$ . As in the classical GOS formula, this ramification-theoretic invariant is related to the Euler characteristic  $\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F})$ :

$$\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F}) = \text{rank}(\mathcal{F}) \cdot \chi_c(\mathfrak{U}_{\overline{K}}, \overline{\mathbb{F}}_\ell) - \sum_{x \in \mathfrak{U}^c \setminus \mathfrak{U}} \text{Sw}_x(\mathcal{F}).$$

See [Hub01, Theorem 10.2]. Here we call it Huber's GOS formula.

Recall the definition of  $\text{Sw}_x(\mathcal{F})$ . Let  $j: \mathfrak{U} \hookrightarrow \mathfrak{U}^c$  be the natural open immersion. Then, the stalk of  $j_*\mathcal{F}$  at  $x$  gives an  $\overline{\mathbb{F}}_\ell$ -representation  $\rho_{\mathcal{F},x}$  of  $G_{\kappa(x)}$ . Take a finite Galois extension  $L/\kappa(x)$  such that  $\rho_{\mathcal{F},x}$  factors through  $\text{Gal}(L/\kappa(x))$ . The valuation  $v_x$  canonically extends to a valuation  $v_x: L^\times \longrightarrow \Gamma_L$  on  $L$ , where  $\Gamma_L \subset \Gamma_x \otimes_{\mathbb{Z}} \mathbb{Q}$  is again naturally isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ .

**Definition 2.1** Let  $\gamma_L$  be the minimal element in  $\{\gamma \in \Gamma_L \mid \gamma > 0\}$  (it corresponds to  $(0,1) \in \mathbb{Z} \times \mathbb{Z}$ ). Let  $\#: \Gamma_L \longrightarrow \mathbb{Z}$  be the map  $\Gamma_L \cong \mathbb{Z} \times \mathbb{Z} \xrightarrow{\text{pr}_2} \mathbb{Z}$ . It is characterized by  $\#v_K(a) = 0$  for  $a \in K^\times$  and  $\#\gamma_L = 1$ .

For  $\sigma \in \text{Gal}(L/\kappa(x))$  with  $\sigma \neq 1$ , put

$$h_{L/\kappa(x)}(\sigma) = \min \left\{ v_x \left( \frac{\sigma(a) - a}{a} \right) \mid a \in L^\times \right\} \in \Gamma_x.$$

It is known that if  $\varpi_L \in L^\times$  satisfies  $v_x(\varpi_L) = \gamma_L$ , then

$$h_{L/\kappa(x)}(\sigma) = v_x \left( \frac{\sigma(\varpi_L) - \varpi_L}{\varpi_L} \right).$$

Define  $s_{L/\kappa(x)}: \text{Gal}(L/\kappa(x)) \longrightarrow \mathbb{Z}$  by

$$s_{L/\kappa(x)}(\sigma) = -\#h_{L/\kappa(x)}(\sigma) \quad (\sigma \neq 1), \quad s_{L/\kappa(x)}(1) = \sum_{\sigma \neq 1} \#h_{L/\kappa(x)}(\sigma).$$

Then, there exists a virtual representation  $\text{Sw}_{L/\kappa(x)}$  of  $\text{Gal}(L/\kappa(x))$  over  $\mathbb{Z}_\ell$  whose character coincides with  $s_{L/\kappa(x)}$  (cf. [Hub01, Theorem 4.1]). Now we put

$$\text{Sw}_x(\mathcal{F}) = \text{Sw}_{\kappa(x)}(\rho_{\mathcal{F},x}) = \dim_{\overline{\mathbb{F}}_\ell} \text{Hom}_{\text{Gal}(L/\kappa(x))}(\text{Sw}_{L/\kappa(x)} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{F}}_\ell, \rho_{\mathcal{F},x}).$$

**Remark 2.2** We can also define the Swan conductor by using the upper ramification filtration of  $G_{\kappa(x)}$  (cf. [Hub01, §2]), as in the classical case.

Huber's GOS formula can also be formulated within the framework of scheme theory. For given rigid curve  $\mathfrak{U}$ , we can always find

- a regular scheme  $X$  which is proper, flat and relatively one-dimensional over  $\mathcal{O}_K$ ,
- and an open subscheme  $U$  of  $X$  such that  $X_s \setminus U_s$  consists of finite points and the special fiber  $X_s$  is smooth at every point in  $X_s \setminus U_s$

such that the rigid generic fiber of the formal completion of  $U$  along  $U_s$  is isomorphic to  $\mathfrak{U}$ . Assume moreover that our  $\overline{\mathbb{F}}_\ell$ -sheaf  $\mathcal{F}$  on  $\mathfrak{U}$  is induced from a smooth  $\overline{\mathbb{F}}_\ell$ -sheaf, also denoted by  $\mathcal{F}$ , on the generic fiber  $U_\eta$  of  $U$ . Then,

- we have  $\chi_c(\mathfrak{U}_{\overline{K}}, \mathcal{F}) = \chi_c(U_s, R\psi\mathcal{F})$ , and
- there exists a canonical bijection  $\mathfrak{U}^c \setminus \mathfrak{U} \cong X_s \setminus U_s$ .

For  $x \in \mathfrak{U}^c \setminus \mathfrak{U} = X_s \setminus U_s$ , the 2-dimensional valuation field  $(\kappa(x), v_x)$  can be described as follows. Let  $Y$  be the unique irreducible component of  $X_s$  containing  $x$  and  $\xi$  the generic point of  $Y$ . Then,  $\xi$  gives a prime ideal  $\mathfrak{p}$  of the henselian local ring  $A = \mathcal{O}_{X,x}^h$ . Let  $V_x$  be the inverse image of  $A/\mathfrak{p}$  under the canonical surjection  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/\mathfrak{p} = \text{Frac } A/\mathfrak{p}$ . It is a two-dimensional valuation ring; indeed,  $A_{\mathfrak{p}}$  and  $A/\mathfrak{p}$  are discrete valuation rings, and the construction above is a well-known process to compose two valuation rings. For an arbitrary lift  $\tilde{t} \in \mathcal{O}_{X,x}$  of a local coordinate  $t$  of  $Y$  at  $x$ , the valuation  $v_x: \text{Frac } A_{\mathfrak{p}} \rightarrow \mathbb{Z} \times \mathbb{Z}$  corresponding to  $V_x$  can be characterized by the formula  $v_x(a\tilde{t}^m) = (v_K(a), m)$ , where  $a \in K$  and  $m \in \mathbb{Z}_{\geq 0}$ . We can identify  $(\text{Frac } A_{\mathfrak{p}}, v_x)$  with the valuation field  $(\kappa(x), v_x)$ .<sup>2</sup> The representation  $\rho_{\mathcal{F},x}$  of  $G_{\kappa(x)}$  is the pull-back of the  $\pi_1(U_\eta)$ -representation attached to  $\mathcal{F}$  under the natural homomorphism  $G_{\kappa(x)} = \pi_1(\text{Frac } A_{\mathfrak{p}}) \rightarrow \pi_1(U)$ .

**Problem 2.3** Consider the case where  $X = \mathbb{P}_{\mathcal{O}_K}^1$  and  $U = \text{Spec } \mathcal{O}_K[T, T^{-1}]$ . For an integer  $m \geq 0$ , we have a covering of  $U$  defined by the equation  $S^m = T$ , which is étale over the generic fiber. This covering and a non-trivial character  $\chi: \mathbb{Z}/m\mathbb{Z} \rightarrow \overline{\mathbb{F}}_\ell^\times$  give rise to a smooth sheaf  $\mathcal{L}_\chi$  of rank 1 over  $U_\eta$ . Compute  $\chi_c(U_s, R\psi\mathcal{L}_\chi)$  by using Huber's GOS formula. More generally, consider the case of a covering defined by the equation  $S^m = f$ , where  $f \in \mathcal{O}_K[T]$ .

**Problem 2.4** Assume that  $K$  contains a primitive  $p$ -th root of unity  $\zeta_p$ . We put  $z = \zeta_p - 1$  and consider a covering of  $\mathbb{A}_{\mathcal{O}_K}^1 = \text{Spec } \mathcal{O}_K[T]$  defined by the equation

$$\frac{(1 - zS)^p - 1}{z^p} = -f,$$

where  $f \in \mathcal{O}_K[T]$  (cf. [KS10, §8.1]). On the generic fiber, it is a Kummer covering of degree  $p$ ; on the other hand, it induces an Artin-Schreier covering  $T^p - T = \bar{f}$  on

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<sup>2</sup>Actually they are slightly different (we need certain completion of  $\text{Frac } A_{\mathfrak{p}}$ ), but it does not affect computations of the Swan conductor.

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the special fiber. Let  $U \subset \mathbb{A}_{\mathcal{O}_K}^1$  be the étale locus of this covering, and  $X = \mathbb{P}_{\mathcal{O}_K}^1$ . This covering and a non-trivial character  $\psi: \mathbb{Z}/p\mathbb{Z} \rightarrow \overline{\mathbb{F}}_\ell^\times$  give a smooth sheaf  $\mathcal{L}_{f,\psi}$  of rank 1 over  $U_\eta$ . Compute  $\chi_c(U_s, R\psi\mathcal{L}_{f,\psi})$  by using Huber's GOS formula.

If the rank of  $\mathcal{F}$  is one,  $\text{Sw}_x(\mathcal{F})$  can be described by means of the class field theory for higher dimensional local fields (*cf.* [Kat87, §4]). This suggests us that  $\text{Sw}_x(\mathcal{F})$  is related to the refined Swan conductor in [Kat89] and [Kat94].

**Problem 2.5** In the cases of Problem 2.3 and Problem 2.4, compute the refined Swan conductor of  $\mathcal{L}_\chi$  and  $\mathcal{L}_{f,\psi}$ , respectively (to do it, [Kat89, §4] and [KS10, §8] will be useful). Find a relationship between  $\text{Sw}_x$  and the refined Swan conductor.

**Problem 2.6** Consider a higher-dimensional generalization of Huber's GOS formula in the rank 1 case by using the refined Swan conductor.

**Problem 2.7** A proof of Huber's GOS formula is based on the Lefschetz trace formula [Hub01, Theorem 6.3]. Instead of this formula, use the log Lefschetz trace formula of Kato-Saito [KS10, §1] to obtain another GOS formula. Note that, in general if  $X$  is a proper strictly semistable  $\mathcal{O}_K$ -scheme and  $U$  is an open subscheme of  $X$  such that  $H = X \setminus U$  is flat over  $\mathcal{O}_K$  and  $(X, X_s \cup H)$  is a strictly semistable pair (*cf.* [dJ96, 6.3]), then the nearby cycle cohomology  $H_c^i(U_s, R\psi\mathbb{Q}_\ell)$  is isomorphic to  $H_c^i(U_{\overline{K}}, \mathbb{Q}_\ell)$ . This makes it possible to apply the log Lefschetz formula to  $H_c^i(U_s, R\psi\mathbb{Q}_\ell)$ .

**Problem 2.8** Try to find a higher-dimensional generalization of the formula developed in Problem 2.7 by using a similar method as in [KS08]. Compare it with the rank 1 case considered in Problem 2.6.

## References

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