DIVISION ALGEBRAS AND PATCHING

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Course outline and project descriptions
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1. Description of the course

This course provides an introduction to the technique of patching with a focus on applications to division algebras.

Patching methods permit the construction and study of global objects through more local objects. The approach to patching that will be presented is that of *patching over fields*, which was developed by the speakers in [HH10]. This approach assumes just linear algebra and basic familiarity with algebraic curves, avoiding the need for formal schemes and rigid analytic spaces. Despite its simplicity, it has a broader range of applications than other approaches; these are not limited to Galois theory, but also include division algebras (see [HHK09] and [HHK11]), quadratic forms, and differential algebra. We will discuss patching from this point of view, presenting the development and proving results in key cases. We will then show how patching can be used to prove results in some of the above mentioned areas of algebra, and in particular we will use it to approach the problems we will discuss concerning division algebras.

One of the problems about division algebras that we will study concerns the notion of admissibility, which was introduced by Schacher in 1968 [Sch68]. A finite group G is said to be *admissible* over a field F if there is a (central) division algebra A over F that contains a G-Galois field extension of F as a maximal subfield. This notion can also be interpreted in terms of crossed product algebras and can be viewed as a division algebra analogue to the inverse Galois problem. Schacher conjectured that a finite group G is admissible over $\mathbb{Q}$ if and only if each Sylow subgroup of G is metacyclic (i.e., an extension of a cyclic group by a cyclic group). He proved the forward direction of this, and showed the converse for abelian groups. Since then, other cases of the converse have been shown by a number of authors, but in general the problem remains open.

In this course, we will consider the analogous question in the case when F is a one-variable function field over a complete discretely valued field (e.g. $\mathbb{C}((t))(x)$ or $\mathbb{Q}_p(x)$). Such fields are amenable to patching methods, where objects over F are studied via the objects they induce over certain overfields F_ξ that are obtained from F via completions. Over such fields F, we will also use patching to treat other problems that are related to division algebras, e.g. finding the Brauer group of F in terms of the Brauer groups of the fields F_ξ, and finding the index of a central simple F-algebra A in terms of those of the induced F_ξ-algebras A_ξ. These latter problems hint at the relationship of patching to local-global principles; such principles are classical over global fields but can also be considered over fields F as above.

In summary, the goal of the course is to provide participants with exposure both to patching and to problems in the theory of division algebras. Participants should have some familiarity in advance with basics concerning algebraic curves and associative algebras; see below for more information about this, and for references for this background material.

2. Content of the lectures

The lectures will provide a presentation of patching over fields, and will use this to study problems in division algebras. The lectures will include the following topics:

Patching over fields: patching problems, matrix factorization, "patching fields" associated to function fields, intersection property, Weierstrass Preparation Theorem, construction of global objects via local ones, applications (including to Galois theory).

Topics on division algebras: central simple algebras and division algebras, basic notions (degree, period, index), Brauer groups, cyclic algebras, crossed products, admissibility, applications of patching to problems concerning division algebras and central simple algebras.

3. Some sample projects

The list below is intended to give an idea of the type of projects that participants will work on. They range from exercises to more difficult questions to open problems.

Problems about patching:

(1) Determine the patching fields explicitly in the case when the base field is $\mathbb{Q}_p(x)$.
(2) Prove the intersection property for patching fields explicitly in the case of $\mathbb{C}((t))(x)$.
(3) Do the same for the matrix factorization property.
(4) Interpret the patching fields in terms of convergent power series.
(5) Prove intersection and matrix factorization for a function field over $\mathbb{C}((s,t))$.
(6) Find a field extension of $F = \mathbb{C}((t))(x)$ that induces given explicit extensions of the fraction fields F_1, F_2 of $\mathbb{C}[x][[t]]$ and of $\mathbb{C}[x^{-1}][[t]]$, compatibly over the fraction field F_0 of $\mathbb{C}[x, x^{-1}][[t]]$.
(7) In the notation of the previous problem, suppose that E/F is a field extension such that EF_i is a Galois extension of F_i for $i = 1, 2$. Must E be Galois over F?
(8) Determine whether the intersection property holds for completions taken with respect to a line in $\mathbb{P}^1 \times \mathbb{P}^1$, in $\mathbb{P}^2$, or in a rational ruled surface.
(9) Show that if Z is a homogeneous space for GL_n over F, and if Z has rational points over each patching field F_ξ for F, then it has an F-rational point.
(10) Can "patching fields" and their overlaps be defined for the field $F = \mathbb{Q}$?

Problems about division algebras:

(1) Give an example of two non-isomorphic division algebras over $\mathbb{Q}$ of the same degree.
(2) Find an example of a cyclic extension F of $\mathbb{Q}$ of degree four such that F is a maximal subfield of a $\mathbb{Q}$-division algebra.
(3) Do the same with respect to the group $V_4 = C_2 \times C_2$, instead of C_4.
(4) Show that this new example necessarily contains a C_4-Galois subfield.
(5) Show by example that if A is a central simple algebra over F, then the index of the induced F_ξ-algebra A_ξ can be strictly less than that of A for some ξ. In fact, can it be strictly less for every ξ?
(6) Construct a division algebra over $F = \mathbb{C}((t))(x)$ that contains a maximal subfield that is Galois over F with group S_3.
(7) Study the classes of groups whose Sylow subgroups are all metacyclic or all abelian of rank two, respectively. In particular, give examples of such groups.

(8) Show by example what goes wrong if one patches together two division algebras over $F = \mathbb{C}((t))(x)$, each having maximal subfields with Galois group $C_2 \times C_2$, in an attempt to prove admissibility of C_2^4 over F.
(9) What can be said about admissible groups over $F = k((t))(x)$ if k is not algebraically closed? What if k has positive characteristic? What if $k((t))$ is replaced by $\mathbb{Q}_p$?
(10) Find explicit examples of admissible groups whose order is divisible by the residue characteristic of our base field.

4. Prerequisites

Participants are expected to be familiar with some basic concepts from algebraic geometry. These include the notions of affine and projective varieties, algebraic curves over a field, the function field of a curve, a divisor on a curve, the spectrum of a commutative ring, and the Zariski topology. This material is contained, for example, in Chapter I (sections 1-6) and Chapter II (sections 1-3) of Hartshorne [Hts77].

Familiarity with basic concepts from commutative ring theory, field theory, and related areas of algebra are also assumed. These include discrete valuation rings, Galois field extensions, integral closure, finitely generated modules, finite groups, and the notion of a category. This material can be found, for example, in Hungerford [Hun74].

Concerning associative algebras, it would be helpful for participants to familiarize themselves beforehand with the notions of central simple algebras, division algebras, matrix algebras, the Brauer group, Wedderburn's structure theorem for central simple algebras, cyclic algebras, and crossed product algebras. For example, see [Her68], especially Chapter 4. Another useful reference is [Pie82].

References

[HH10] David Harbater and Julia Hartmann. Patching over fields. Israel J. Math. **176** (2010), 61–107.

[HHK09] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. Invent. Math. **178** (2009), 231–263.

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[Hts77] Robin Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics. Springer-Verlag, 1977.

[Her68] Israel N. Herstein. Noncommutative rings. Math. Assoc. of America, 1968.

[Hun74] Thomas W. Hungerford. Algebra. Graduate Texts in Mathematics. Springer-Verlag, 1974.

[Pie82] Richard S. Pierce. Associative algebras. Springer-Verlag, New York, 1982.

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